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**(A) Higher spin and amplitude methods** [8, 6, 7, 2]

The coupling of a photon of momentum $q$ to a particle of spin S and mass $m$ is given by the 3pt amplitude

$$\mathcal{A}_{+S_AS_B} = Q\frac{\langle AB\rangle^{2S}}{m^{2S-1}}\frac{m[1\xi]}{\langle q\hat{p}_A\xi]}\,, \qquad \mathcal{A}_{-S_AS_B} = Q\frac{[AB]^{2S}}{m^{2S-1}}\frac{m\langle 1\xi\rangle}{\langle q\hat{p}_B\xi]}\,, \tag{1}$$

where $\pm$ are the photon helicities and the spin of particles $A, B$ is encoded in the implicit index of square and angle brackets $|A_I\rangle$, $I = \pm 1/2$.

1. The coupling of an electron and a positive helicity photon in QFT reads,

$$e\begin{pmatrix} [B|\,, & \langle B| \end{pmatrix}\gamma^0\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} |A\rangle \\ |A] \end{pmatrix}\frac{\langle\xi|\sigma^\mu|q]}{\sqrt{2}\langle\xi\, q\rangle}(2\pi)^4\delta^4(p_A + p_B + q)\,.$$

   Work the expression above into angle and square bracket products, then divide and multiply by $\langle q\hat{p}_A\xi]$ simplifying the numerator and using relations

$$\langle\xi|\sigma^\mu|q]\sigma_\mu = |\xi\rangle[q| \qquad \langle\xi|\sigma^\mu|q]\bar{\sigma}_\mu = |q]\langle\xi| \qquad \hat{q}\bar{\xi} + \hat{\xi}\bar{q} = \langle q\xi\rangle[\xi q]\mathbf{1}$$

$$\hat{p}_i|i] = m_i|i\rangle \qquad p_A^\mu q_\mu = 0 \to \hat{p}_A\bar{q} = -\hat{q}\bar{p}_A \qquad p_A + p_B + q = 0$$

   with $m_A = m_B = m$, $m_\gamma = m_1 = 0$ to recover the form in (1) with $Q = e/\sqrt{2}$.

2. Show that the product which arises in the s-channel on shell contribution (i.e. $((p_A + p_1)^2 = (p_C + p_2)^2 = m^2)$) is gauge (i.e. $\xi, \zeta$) independent and simplifies to

$$\frac{m[1\xi]}{\langle 1\hat{p}_A\xi]}\frac{m\langle 2\zeta\rangle}{\langle\zeta\hat{p}_C 2]} = \frac{\langle 2\hat{p}_A 1]^2}{m^2\langle 12\rangle[12]}$$

   using relations given in exercise 1 above with substitutions $q \to p_1, A \to A$ and separately on the other vertex $q \to p_2, A \to C$, $\xi \to \zeta$.

3. In the lectures we derived the high energy behaviour of the s-channel contribution to Compton scattering

$$\mathcal{A}_{+-} = Q^2\frac{m^2 t^S}{s m^{2S}}$$

   unitarity demands, roughly, $\mathcal{A}(t \to s) \leq 16\pi$ (if you want it rigorous, here [8]). Obtain and discuss an estimate for the scale $E_* = \sqrt{s_*} = L_*^{-1}$ at which the bound is saturated and we expect to see the composite nature of higher spin for these particles ($\hbar c = 0.2$ GeV fm, $e^2/(4\pi) = 1/137$)

| | $\Delta$ (Baryon) | $a_2$ (Meson) | $^{115}_{45}$In (Nucleus) |
|---|---|---|---|
| mass(GeV) | 1.2 | 1.3 | 107 |
| spin | 3/2 | 2 | 9/2 |
| charge | $e$ | $e$ | guess |

**(B) Conserved magnitudes from Poincare symmetry** [10, 9]

The infinitesimal effect on a scalar function (e.g. a scalar field and the Lagrangian) under translations $x^\mu \to x^\mu + \epsilon^\mu$, is $\delta_\epsilon f(x) = \epsilon^\mu \partial_\mu f$. Defining $\delta_\epsilon \mathcal{L} = \partial_\mu F^\mu$ we obtain an expression for the conserved current of translations

$$\epsilon^\nu J^\mu_{(\nu)} = \delta_\epsilon \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - F^\mu = \epsilon^\nu T^\mu{}_\nu$$

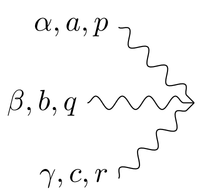
with conserved magnitudes $\int dx^3 J^0_{(\mu)} = \int dx^3 T^0_\mu = P_\mu$ i.e. total energy and momentum.

1. Obtain the currents for Lorentz invariance $x^\mu \to x^\mu + \omega^{\mu\nu} x_\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and express them in terms of $T_{\mu\nu}$ and $x^\mu$.
2. Interpret the conserved magnitudes, first for rotations, then for boosts.

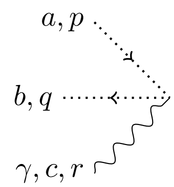
**(C) Running of non-abelian coupling** [1]

The Feynman rules in the background and Feynman gauge are (all momenta coming into the vertex)

$$g f_{abc} \begin{pmatrix} \eta_{\alpha\gamma}(p-r-q)_\beta \\ +\eta_{\beta\alpha}(q-p+r)_\gamma \\ +\eta_{\gamma\beta}(r-q)_\alpha \end{pmatrix}$$




$$g f_{abc}(p-q)_\gamma$$




while the gauge propagator is $-i\eta_{\mu\nu}\delta_{ab}/q^2$ and that of the ghosts $i\delta_{ab}/q^2$.

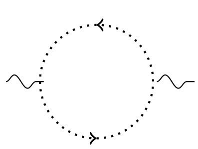
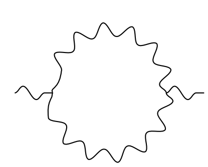
1. Consider the gauge propagator between two *conserved* currents and a self energy correction to it the form $\Sigma_{\mu\nu} = \Sigma_T q^2 \eta_{\mu\nu} + \Sigma_L q_\mu q_\nu$. Justify the dropping of $\Sigma_L$ in the following equation

$$gJ_\mu \left( \frac{-i\eta^{\mu\nu}}{q^2} + \frac{-i\eta^{\mu\rho}}{q^2}(-i\Sigma_{\rho\sigma})\frac{-i\eta^{\sigma\nu}}{q^2} \right) gJ_\nu = gJ_\mu \left( \frac{-i\eta^{\mu\nu}}{q^2} + \frac{-i\eta^{\mu\nu}}{q^2}(-i\Sigma_T q^2)\frac{-i}{q^2} \right) gJ_\nu$$

2. Using Feynman parameters and the results

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell_\mu \ell_\nu}{(\ell^2 - \Delta)^2} = \frac{1}{2}\frac{1}{(4\pi)^2}\Delta\, d_\epsilon \qquad \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(4\pi)^2} d_\epsilon$$

given $d_\epsilon \equiv \epsilon^{-1} - \log p^2/\mu^2$ with $p$ the momentum in $\Delta$, compute ($P_{\mu\nu} = q^2\eta_{\mu\nu} - q_\mu q_\nu$)

$-i\Sigma_{\text{ghost}} =$  $= \dfrac{iC_{Ad} g_s^4}{3(4\pi)^2}\delta_{ab} P_{\mu\nu} d_\epsilon \quad -i\Sigma_{\text{gauge}} =$  $= \dfrac{i10 C_{Ad} g_s^2}{3(4\pi)^2}\delta_{ab} P_{\mu\nu} d_\epsilon$

where $C_{Ad}\delta_{ab} = f_{acd} f_{bcd}$, the gauge loop has a 1/2 symmetry factor and the ghost an extra minus sign. Why do we get the precise combination in $P_{\mu\nu}$?

3. Put this result in the expression of 1 with the renormalised ($A = \sqrt{Z} A_R$ $Z = 1 + \delta Z$), 2pt function

$$-iJ_\mu \frac{1}{Zq^2} g^2 (1 - \Sigma_T) J_\mu = -iJ_\mu \frac{1}{q^2} g^2 (1 - \Sigma_T - \delta Z) J_\mu + \mathcal{O}(g^6) \equiv -iJ_\mu \frac{g^2_{\text{eff}}}{q^2} J^\mu$$

Use $\delta Z \propto 1/\epsilon$ to cancel the divergence. This being done, how does the effective coupling change with energy ($p$)? You can think of QCD with $C_{Ad} = N_c = 3$ and compare with the $\beta$ function to check your result.

4. If you're up for it add $N_f$ Dirac fermions (quarks) to find $-i\Sigma_\psi = ig^2 4C_\psi \delta P_{\mu\nu} d_\epsilon / 3(4\pi)^2$.

**(D) Self-consistent gauge theories**

Consider the matter content of the SM with variable representations

| | $q_L$ | $u_R$ | $d_R$ | $\ell_L$ | $e_R$ |
|---|---|---|---|---|---|
| $U(1)_Y$ | $Q_q$ | $Q_u$ | $Q_d$ | $Q_\ell$ | $-1$ |
| $SU(2)_L$ | $n$ | $1$ | $1$ | $2$ | $1$ |
| $SU(3)_c$ | $r$ | $r$ | $r$ | $1$ | $1$ |

where $n$, $r$ label the representations by their dimension, i.e. $n = 2, 3, 4$, $r = 3, 6, 8$ etc.

1. Revise anomaly cancellation (including gravity$\times$hypercharge) in this theory to obtain the constraints:

$$Q_u + Q_d - nQ_q = 0 \qquad r(Q_u + Q_d - nQ_q) - 1 - 2Q_\ell = 0 \qquad (2 - n) = 0$$

$$rC_nQ_q + C_2Q_\ell = 0 \qquad r(Q_u^3 + Q_d^3 - nQ_q^3) - 1 - 2Q_\ell^3 = 0$$

where $C_n$ are Casimirs $\mathrm{tr}(T^a_{(n)}T^b_{(n)}) \equiv C_n\delta_{ab}$.

2. Solve for hypecharges as a function of $r$ to find

| $Q_q$ | $Q_u$ | $Q_d$ | $Q_\ell$ | $Q_{e_R}$ |
|---|---|---|---|---|
| $\frac{1}{2r}$ | $\frac{1}{2r} + \frac{1}{2}$ | $\frac{1}{2r} - \frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ |

3. Take the symmetric representation $r = 6$ built out of two symmetrised fundamentals and give the electric charges of pions, protons and neutrons in this theory ($Q_{\rm em} = Q_Y + T_3$, with $T_3 = \sigma_3/2$ in the $SU(2)$ fundamental).

**(E) Non-invertible symmetry and pion decay**[5, 3, 4]

Consider the insertion of the charge operator at $t = 0$ for the conserved gauge-invariant non-invertible symmetry found in [5] for the neutral pion action

$$S = \int_{t>0} d^4x \left( \frac{1}{2}\partial_\mu\pi_0\partial^\mu\pi_0 + \frac{c_A\pi_0}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \right) + \int_{t=0} d^3x \left( \frac{\pi}{N}J_A^0 + \frac{N}{4\pi b_A}a^i\epsilon_{ijk}\left[\frac{a^{jk}}{2} - \frac{b_AF^{jk}}{N}\right] \right)$$
$$+ \int_{t<0} d^4x \left( \frac{1}{2}\partial_\mu\pi_0\partial^\mu\pi_0 + \frac{c_A\pi_0}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \right)$$

where $\pi_0(t \to 0^+) = \pi_0(t \to 0^-) - 2\pi f_\pi/N$, $b_A = N_c(q_u^2 - q_d^2)$, $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, $a^{ij} = \partial^i a^j - \partial^j a^i$ while every other field is the same above and below $t = 0$.

1. Use the variational principle to compute the EoM for $A_\mu \to A_\mu + \delta A_\mu$ with care to account for boundary terms and show that these read

$$\int_{t=0} d^3x \left[\delta A_\mu c_A[\pi_0(0^-) - \pi_0(0^+)]\tilde{F}^{0\mu} + \frac{1}{2\pi}\epsilon_{ijk}\partial^j a^k\delta A^i\right] = \int_{t=0} d^3x\delta A_i\epsilon^{ijk}\left[c_A\frac{2\pi f_\pi}{N}F^{jk} - \frac{1}{4\pi}a^{jk}\right]$$

2. Combine the equation above with the EoM for $a$ and show that consistency demands

$$c_A = \frac{N_c(q_u^2 - q_d^2)}{8\pi^2 f_\pi}$$

# References


[1] L. F. Abbott. Introduction to the Background Field Method. *Acta Phys. Polon. B*, 13:33, 1982.

[2] Nima Arkani-Hamed, Tzu-Chen Huang, and Yu-tin Huang. Scattering amplitudes for all masses and spins. *JHEP*, 11:070, 2021.

[3] Lakshya Bhardwaj, Lea E. Bottini, Ludovic Fraser-Taliente, Liam Gladden, Dewi S. W. Gould, Arthur Platschorre, and Hannah Tillim. Lectures on generalized symmetries. *Phys. Rept.*, 1051:1–87, 2024.

[4] T. Daniel Brennan and Sungwoo Hong. Introduction to Generalized Global Symmetries in QFT and Particle Physics. 6 2023.

[5] Yichul Choi, Ho Tat Lam, and Shu-Heng Shao. Noninvertible Global Symmetries in the Standard Model. *Phys. Rev. Lett.*, 129(16):161601, 2022.

[6] Herbi K. Dreiner, Howard E. Haber, and Stephen P. Martin. Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry. *Phys. Rept.*, 494:1–196, 2010.

[7] Henriette Elvang and Yu-tin Huang. Scattering Amplitudes. 8 2013.

[8] M. Jacob and G.C. Wick. On the general theory of collisions for particles with spin. *Annals of Physics*, 7(4):404–428, 1959.

[9] R.A. https://www.ippp.dur.ac.uk/ ralonso/iqft.pdf. *My QFT*, notes.

[10] David Tong. http:/www.damtp.cam.ac.uk/user/tong/qft.html. *QFT*, Lectures.