# Introduction to particle physics
# Lecture 9: Gauge invariance

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# Outline

# Symmetries in classical physics

## Invariance and conservation laws

- From classical physics it is known that invariance of a system under certain transformations is related to the conservation of corresponding quantities.
- Examples:

| Invariance under | | Conserved quantity |
|---|---|---|
| rotations | $\Longleftrightarrow$ | angular momentum |
| time translations | $\Longleftrightarrow$ | energy |
| space translations | $\Longleftrightarrow$ | momentum |

- Formalised by Emmy Noether (thus: Noether's theorem)

## Invariance and conservation laws

- The same ideas work also in quantum physics:
  Invariances give rise to conservation laws.
- There, however, internal symmetries also play a role.
  In fact, they are used to construct interactions in theories.
- Example in particle physics:
  - Invariance under phase transformations of the fields
    $$\psi(x,\, t) \to \psi'(x,\, t) = \exp(i\theta)\psi(x,\, t) \iff |\psi|^2 = |\psi'|^2$$
    yields conserved charges like, e.g., the electrical charge.
  - Note: global changes in phase cannot be observed (because typically squares are taken), but phase differences are observable.
    (In Quanutm Mechanics: Aharonov-Bohm effect.)
  - The photon field couples to this charge and is thus related to the invariance under such phase transitions (later more). (Will come to that later.)

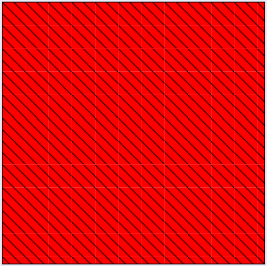
## Mathematical formulation

(Not examinable)

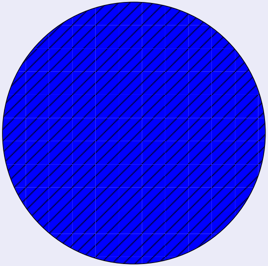
- Ideas of symmetry are formalised in group theory.
- Definition of groups:
    - Consider sets of elements $\mathcal{S} = \{a,\ b,\ \ldots\}$ with operation "$\cdot$": $a \cdot b$.
    - Such sets are called groups, if
      $a \cdot b \in \mathcal{S}$ (closure)
      $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity)
      $\exists 1 \in \mathcal{S}$: $a \cdot 1 = 1 \cdot a = a\ \forall a \in \mathcal{S}$ (neutral element)
      $\forall a \in \mathcal{S}$: $\exists a^{-1} \in \mathcal{S}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ (inverse element).
- Examples: $\mathcal{S}$ = integer numbers, $\cdot = +$, rotations with arbitrary angles, the set $\{1, 2, 3, \ldots, p-1\}$ under multiplication modulo $p$, if $p$ is a prime number, ...

## Discrete vs. continuous symmetries

- Consider two slabs with quadratic and round cross section.
- The quadratic one has a discrete symmetry w.r.t. rotation along its axis, while the round one enjoys a continuous symmetry.



only multiples of 90 degrees



all angles

- More physical examples: parity vs. angular momenutm

# Classical gauge invariance

## Fields and potentials in electrodynamics

- Remember Maxwells equation:

$$\begin{aligned} \underline{\nabla} \cdot \underline{E} &= 4\pi\rho & \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= 0 & \underline{\nabla} \times \underline{B} - \frac{\partial \underline{E}}{\partial t} &= 4\pi \underline{j}\,. \end{aligned}$$

- Implicit: conservation of current, $\dot{\rho} + \underline{\nabla} \cdot \underline{j} = 0$.
- Can introduce potentials $\Phi$ and $\underline{A}$ such that

$$\underline{E} = -\underline{\nabla}\Phi - \frac{\partial \underline{A}}{\partial t} \quad \text{and} \quad \underline{B} = \underline{\nabla} \times \underline{A}\,.$$

(Can read them off from homogenous equations, i.e. equations of the form l.h.s.=0.)

- Gauge invariance: The electromagnetic fields will not change under

$$\Phi \Longrightarrow \Phi' = \Phi + \frac{\partial \Lambda}{\partial t} \quad \text{and} \quad \underline{A} \Longrightarrow \underline{A}' = \underline{A} - \underline{\nabla}\Lambda$$

(This is the gauge transformation of classical electrodynamics with an **arbitrary scalar function** $\Lambda$.)

## Lorentz force

- Lorentz-force reads:

$$\underline{F} = e\left[\underline{E} + \frac{\mathrm{d}\underline{x}}{\mathrm{d}t} \times \underline{B}\right] = e\left[-\underline{\nabla}\Phi - \frac{\partial \underline{A}}{\partial t} + \frac{\mathrm{d}\underline{x}}{\mathrm{d}t} \times \underline{\nabla} \times \underline{A}\right]$$
$$= e\left[-\underline{\nabla}\Phi - \frac{\mathrm{d}\underline{A}}{\mathrm{d}t} + \underline{\nabla} \cdot \left(\frac{\mathrm{d}\underline{x}}{\mathrm{d}t} \cdot \underline{A}\right)\right] .$$

To see this, use that $\frac{\mathrm{d}A_x}{\mathrm{d}t} = \frac{\partial A_x}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial A_x}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial A_x}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial A_x}{\partial z} = \frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}$

and that $(\underline{v} \times \underline{\nabla} \times \underline{A})_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + v_x\left(\frac{\partial A_x}{\partial x} - \frac{\partial A_x}{\partial x}\right)$

and that, since $\frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial x} = \frac{\partial v_z}{\partial x} = 0$, $\underline{v} \cdot (\underline{\nabla} \cdot \underline{A}) = \underline{\nabla}(\underline{v} \cdot \underline{A})$

- This can be used to construct a Lagrange function, rederive E.o.M. with Euler-Lagrange method & confirm the force, assess symmetries, construct a Hamilton function to handle the quantum mechanical problem . . . .

## Lagrange and Hamilton function

- Therefore: Lagrange function is given by

$$L = \frac{m}{2}\left(\frac{\mathrm{d}\underline{x}}{\mathrm{d}t}\right)^2 - e\left(\Phi + \frac{\mathrm{d}\underline{x}}{\mathrm{d}t}\cdot\underline{A}\right)$$

and the Hamilton function reads

$$H = \frac{1}{2m}\left(\underline{p} - e\underline{A}\right)^2 + e\Phi\,.$$

- Can use this to generalise for all situations:
  To treat interactions of a particle with a field, replace the momentum with the generalised one,

$$\underline{p} \longrightarrow \underline{\Pi} = \underline{p} - e\underline{A}.$$

- I will argue that this can be enforced by a gauge principle.

# Phase invariance

## Phase invariance in Quantum Mechanics

- Consider a system in a state $|\psi(t)\rangle$, alternatively described by a wave function $\psi(t, \underline{x}) = \langle \underline{x}|\psi(t)\rangle$.
- Remember that probabilities are related to $\langle\psi(t)|\psi(t)\rangle$ or $|\psi(t, \underline{x})|^2$.
- Clearly, one can redefine

$$\begin{aligned} |\psi(t)\rangle &\longrightarrow |\psi(t)\rangle' = e^{-i\alpha}|\psi(t)\rangle \\ \psi(t, \underline{x}) &\longrightarrow \psi(t, \underline{x})' = e^{-i\alpha}\psi(t, \underline{x}) \,. \end{aligned}$$

  without changing measurements related to $\langle\psi(t)|\psi(t)\rangle$ or $|\psi(t, \underline{x})|^2$.
- This phase invariance is a simple example of a continous symmetry, described by a continous parameter, here the phase $\alpha$.
- Symmetries are described by groups, represented by matrices. The group related to this symmetry is called $U(1)$, the group of unitary matrices of dimension 1 (complex numbers with absolute value 1).

## QED: Global phase invariance

(Details of equations not examinable)

- Lagrange formulation with fields:
  - generalised coordinates $q(t) \longrightarrow$ fields $\phi(t, \underline{x})$.
  - Lagrange function $L(q, \dot{q}) \longrightarrow$ Lagrangian density $\mathcal{L}(\phi(x_\mu), \partial_\mu(\phi)$ with $L = \int \mathrm{d}^3x\mathcal{L}$.
- Example: Free complex scalar field(s) $\phi$ and $\phi^*$ with mass $m$:

$$\mathcal{L} = (\partial_t\phi)(\partial_t\phi^*) - \underline{\nabla}\phi \cdot \underline{\nabla}\phi^* - m^2\phi\phi^* .$$

- Lagrange function (and with it E.o.M.) invariant under transformation $\mathcal{G}$: $\mathcal{G}\mathcal{L}(\phi, \phi^*) = \mathcal{L}(\phi, \phi^*)$

$$\mathcal{G}\phi = \phi' = e^{-i\alpha}\phi \quad \text{and} \quad \mathcal{G}\phi^* = \phi'^* = e^{i\alpha}\phi^* .$$

- This yields conserved charges $\pm 1$ (group $\mathcal{G}$ again $U(1)$).
- Since the transformation acts the same way on all points in space-time, a symmetry transformation like this is called **global**.

## Local phase transformations

- To measure phase differences: Must establish a specific $\theta = 0$.
- Different conventions related by global phase transformations.
  Clearly the choice, being unobservable, must not matter for physical observables:
  The theory must be invariant under global phase transformations.
- What happens if the phase depends on space-time: $\theta \to \theta(t, \underline{x})$?
  (This is called a local phase transformation.)
  Simple answer: Then the Lagrangian is not invariant any more.

$$\mathbf{G}(x)\mathcal{L}(\phi) \to \mathcal{L}'(\phi) \neq \mathcal{L}(\phi).$$

$$\begin{aligned}
\mathcal{L}' &= [e^{-i\alpha}(-i\partial_t\alpha \cdot \phi + \partial_t\phi)][e^{+i\alpha}(+i\partial_t\alpha \cdot \phi^* + \partial_t\phi^*)] \\
&\quad -[e^{-i\alpha}(-i\underline{\nabla}\alpha \cdot \phi + \underline{\nabla}\phi)] \cdot [e^{+i\alpha}(+i\underline{\nabla}\alpha \cdot \phi^* + \underline{\nabla}\phi^*)] - m^2\phi\phi^* \\
&= \mathcal{L} + (i\partial_t\alpha)\left(\phi^*\partial_t\phi - \phi\partial_t\phi^*\right) - (i\underline{\nabla}\alpha) \cdot \left(\phi^*\underline{\nabla}\phi - \phi\underline{\nabla}\phi^*\right) + \left[(\partial_t\alpha)^2 - (\underline{\nabla}\alpha)^2\right]\phi\phi^* .
\end{aligned}$$

## Restoration of local phase invariance

- But: Can make the Lagrangian invariant by introducing another field, $A$.

$$\mathbf{G}(x)\mathcal{L}(\phi, \phi^*, A) \to \mathcal{L}(\phi', \phi'^*, A').$$

- Properties of this field:
  - Must be massless to allow for infinite range - it must connect different phase conventions all over space.
  - It's a four vector, $A^\mu$, identified with the photon field.
  - The photon field must also transform under $\mathbf{G}(x)$ such that the combination with changes due to the electron field are compensated.
- Couple it with the replacement (from Lorentz force) $p^\mu = (E, \underline{p}) \to \Pi^\mu = (p^\mu - eA^\mu) = (E - e\Phi, \underline{p} - e\underline{A})$
- Summary of this construction:
  - Global phase invariance yields conserved charges.
  - Local phase invariance gives rise to the photon field, i.e. interactions.
- Final remark: A trivial mass term for the photon would look like $\mathcal{L}_m \propto m^2 A^2$ and it is not invariant under local phase transformations.

## Relation to gauge invariance in classical theory

- Write the photon field as $A^\mu = (\Phi, \vec{A})$, where
$$\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A} \text{ and } \vec{B} = \vec{\nabla} \times \vec{A}$$
gives the relation to the electric and magnetic field.
- The potentials are invariant under the **gauge transformation**
$$\Phi \to \Phi' = \Phi + \partial_t \Lambda \text{ and } \vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\Lambda,$$
or, in four-vector notation $(\partial_\mu = (\partial_t, -\vec{\nabla}))$
$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda.$$
- Invariance of the Lagrangian $\mathcal{L} = \vec{E}^2 - \vec{B}^2$ follows trivially.
- Finally: The $\Lambda(x)$ here is more or less identical with the $\theta(x)$ of the local gauge transformation before.

# Generalised gauge invariance

## More complicated symmetries

- Obviously, this can be extended by making the state vector/wave function a vector with components labelled by $j \in [1, N]$, such that

$$|\psi(t, \underline{x})|^2 = \sum_j |\psi(t, \underline{x})_j|^2$$

- Generalised phase transformation assumes $N \times N$-matrix character,

$$\begin{aligned} |\psi(t)\rangle_j &\longrightarrow |\psi(t)\rangle'_k = \left[e^{-i\alpha}\right]_{kj} |\psi(t)\rangle_j \\ \psi(t, \underline{x})_j &\longrightarrow \psi(t, \underline{x})'_k = \left[e^{-i\alpha}\right]_{kj} \psi(t, \underline{x})_j \,. \end{aligned}$$

- Can use base matrices (generators) $T^a_{kj}$ such that

$$\left[e^{-i\alpha}\right]_{kj} = e^{-i\sum_\alpha \alpha^a T^a_{kj}}\,, \quad \alpha^a \in \mathbf{R}\,.$$

Information about the allowed transformations contained in the form and properties of the $N \times N$ base matrices $T^a_{kj}$.

- Prominent examples $SO(N)$, $SU(N)$:

## A "classical" example

- Seemingly, gauge invariance an elegant way to produce interactions.
- Added benefit: protects high-energy behaviour of QED.

  (renormalisability)

- Extent this to other interactions, e.g. of nucleons with pions:
  - Pions transform nucleons into nucleons, put $p$ and $n$ into iso-doublet.

    (isospin: like the spin-up and down states of a fermion)
  - Then: Need gauge transformations acting on the nucleon field $N = (p, n)$, mixing the states.
  - $G(x)$ must have $2 \times 2$ matrix form $\Longrightarrow$ use Pauli matrices as basis: There are 3 Pauli matrices - each corresponds to a field: 3 $\rho$'s!

    Due to Gell-Mann-Nishijima formula: $\rho$'s carry isospin $\Longrightarrow$ self-interactions!
  - Can show that Lagrangian is invariant under global $SU(2)$:

$$\mathbf{G}^{SU(2)}\mathcal{L}(N) \to \mathcal{L}(N').$$

  - But: $\rho$'s not elementary and $\pi$'s are the "true" isospin force carriers.

## Summary

- Introduced the concept of symmetries and their role in creating interactions.