Introduction to particle physics
Lecture 7: Perturbation Theory

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Outline

Perturbative expansion in Quantum Mechanics

Basic idea

- Want to calculate cross section/scattering amplitude $f(\Omega)$ in a complicated potential.
- Remember: $f(\underline{k}, \underline{k}') \propto \langle \underline{k}' | \mathcal{H} | \underline{k} \rangle$
- Sometimes (in fact, in most realistic/interesting cases) exact solution with full Hamiltonian inaccessible.
- Write Hamiltonian as

$$\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{V}$$

  and expand in small parameter λ – this works if solution for "unperturbed Hamiltonian" $\mathcal{H}_0$ is known.

Time-evolution operator

- Consider Schrödinger equation for a state vector $|\psi(t)\rangle$ in the Schrödinger picture: (time-dependent states, time-independent operators)

$$\frac{i\partial}{\partial t}|\psi(t)\rangle = \hat{\mathcal{H}}|\psi(t)\rangle \quad \Longrightarrow \quad |\psi(t)\rangle = \exp\left[-i\hat{\mathcal{H}}(t-t_0)\right]|\psi(t_0)\rangle$$

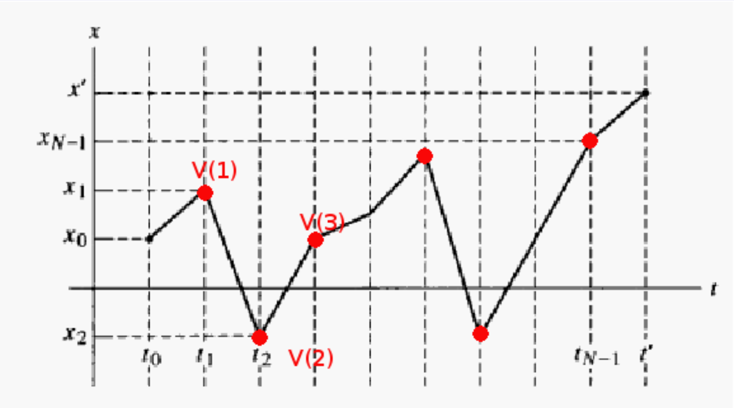
- Temporal evolution of state vector $|\psi(t)\rangle$ through the Hermitian time-evolution operator

$$\hat{\mathcal{U}}(t,\, t_0) = \exp\left[-i\hat{\mathcal{H}}(t-t_0)\right]$$

- Important properties:
  - Identity: $\hat{\mathcal{U}}(t_0,\, t_0) = \hat{\mathbf{1}}$;
  - Composition: $\hat{\mathcal{U}}(t_2,\, t_0) = \hat{\mathcal{U}}(t_2,\, t_1)\hat{\mathcal{U}}(t_1,\, t_0)$.
- Perturbative expansion (with $\hat{\mathcal{U}}_0(t,\, t_0)$ corresponding to $\mathcal{H}_0$):

$$\begin{aligned}\hat{\mathcal{U}}(t,\, t_0) = {} & \hat{\mathcal{U}}_0(t,\, t_0) \\ & + \hat{\mathcal{U}}_0(t,\, t_1)\left[-i\lambda\hat{\mathcal{V}}(t_1)\right]\hat{\mathcal{U}}_0(t_1,\, t_0) \\ & + \hat{\mathcal{U}}_0(t,\, t_2)\left[-i\lambda\hat{\mathcal{V}}(t_2)\right]\hat{\mathcal{U}}_0(t_2,\, t_1)\left[-i\lambda\hat{\mathcal{V}}(t_1)\right]\hat{\mathcal{U}}_0(t_1,\, t_0)\ldots\,,\end{aligned}$$

Pictorial representation of $\hat{\mathcal{U}}(t', t_0)$



Perturbative Expansions in Quantum Field Theory (QFT)

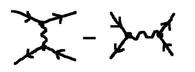
Basic idea

- Relativistic version of QM, represents particles as fields (functions of position x - quantised in QM - and time t).
- Wave functions of, say, individual electrons are excitations of the electron field with a given frequency and wave vector. Summing over these excitations in Fourier space yields the field.
- While "**first quantisation**" recognises the wave nature of particles and the particle nature of waves, this "**second quantisation**" allows for the presence of anti-particles and, accordingly, the possibility to create and annihilate particles.
- Interpretation: Fields can be thought of as harmonic oscillators filling the entire space, one at each position.

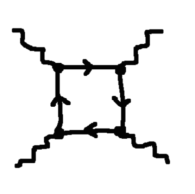
Perturbative expansion

- Replace the potential from above with **interactions between particles**
- Basic idea: Particles as carriers of force (photon carrier of electromagnetic force)
- Coupling constants in interactions parametrise interaction strengths and act as small perturbation parameter λ.
- Assuming the interaction strength between particles is small, **transition amplitudes** $\mathcal{M}$ between particle states can be computed perturbatively.
- Each term of the perturbative amplitude can be represented graphically: **Feynman diagrams**

Interference of amplitudes in $e^- e^+$-scattering



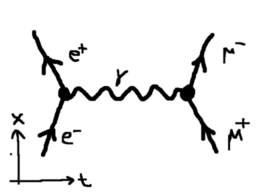
A QM effect: Light-by-light scattering



Virtual particles

- Recall Heisenberg's uncertainty relation: $\Delta E \Delta t \geq 1$.
- In scattering, this allows to create an unphysical particle with a lifetime $\tau \simeq \Delta t \leq 1/\Delta E$

  (See right, the photon is called virtual.)



- No problem, if (local) conservation of energy-momentum guaranteed.
- Such processes are known as a **virtual processes**.
  They typically form the intermediate states of scattering amplitudes, i.e. Feynman diagrams.
- But: Can also "borrow" energy from the vacuum for a time $\tau \simeq 1/\Delta E \longrightarrow$ vacuum fluctuations.

Virtual particles: Quantifying the example

- Consider the reaction $e^+e^- \to \gamma \to \mu^+\mu^-$.

$$E_{e^+e^-} = E_+ + E_- = \sqrt{m^2 + \vec{p}_+^2} + \sqrt{m^2 + \vec{p}_-^2} \geq 0.$$

  Energy-momentum conservation ensures that

$$E_\gamma = E_{e^+e^-} \quad \text{and} \quad \vec{p}_\gamma = \vec{p}_+ + \vec{p}_-.$$

  However, due to the electron's rest mass it is impossible to satisfy

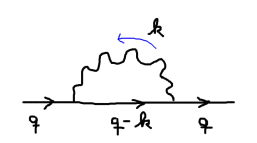
$$E_\gamma^2 - \vec{p}_\gamma^2 = m_\gamma^2 = 0,$$

  the photon is off its mass shell! ($E^2 - p^2 \neq 0! \longrightarrow$ unphysical)
- This implies that the lifetime of the photon is limited: $\Delta t < 1/\Delta E$ in the centre-of-mass frame of the photon ($\vec{p}_\gamma = 0$)
  For a photon with $E_{c.m.} = 200 MeV$, $\tau \leq 1\text{fm}/c \approx 10^{-24}\text{s}$.
- The photon cannot be observed at all - they remain intermediate.

Renormalisation: Sketching the problem

(Not examinable)

- Virtual particles also emerge by, e.g., an electron emitting and re-absorbing a photon.
- Then the four-momenta of the intermediate particles is not fixed by energy-momentum conservation: an integration over the four-momentum inside the loop becomes mandatory.



- In the case above, this quantum correction is related to the integral

$$\int_0^\infty \mathrm{d}^4 k \, \frac{k}{k^2((q-k)^2-m^2)}$$

and diverges - naively linearly.

Renormalisation: Dealing with infinities

(Not examinable)

- The infinities stemming from diagrams like the one above are cured by **redefining** the fields and their interaction strengths to include the quantum corrections, **renormalisation**.
- This is done by adding counter-terms to the theory, which have exactly the same divergence structure.
- In so doing, the quantities in the theory are replaced by “bare” quantities, including all diagrams, including the counter-terms yields finite, physical results.
- The beauty of this concept is that it can be proven to be in principle mathematically well-defined and without ambiguities.
- The catch is, though, that in practical calculations the perturbation series is truncated, leading to residual ambiguities (see later).

Experimental evidence: Running couplings

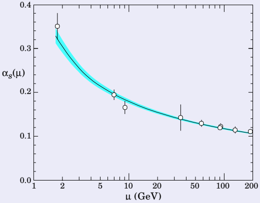
- On of the manifestations of the quantum corrections above is that the couplings (interaction strengths) become scale-dependent!

  (Dependence would vanish, if all perturbative orders calculated.)

- This comes from calculating an observable to a given perturbative order and comparing the result with experiment to extract the coupling strength.
- In particle physics, scales are given in units of energy (inverse lengths).
- Similarly, also masses vary with scale.

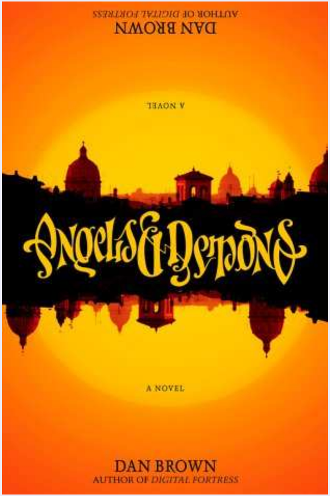
Strong coupling strength

$\alpha_s = g_s^2/(4\pi)$



Anti-matter

. . . and some misconceptions



Reality: Merging special relativity and quantum mechanics

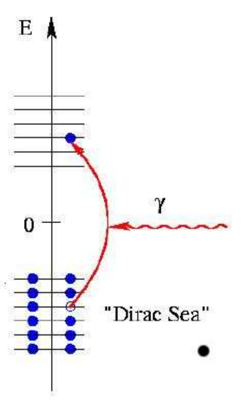
- The Schrödinger equation is non-relativistic, predicting the correct Newtonian relationship between energy and momentum for a particle described by ψ (identifying $E = i\partial/\partial t$ and $p_x = i\partial/\partial x$):
$$E = \frac{\vec{p}^2}{2m} + V.$$
- But for a Lorentz-invariant description, must fulfil the relativistic relation of energy and momentum
$$E^2 = \vec{p}^2 + m^2 \qquad \text{(for a free particle).}$$
- This leads to a **quadratic** equation in E (or $\partial/\partial t$) with positive and negative energies (or advanced and retarded waves) as solutions:
$$E = \pm\sqrt{\vec{p}^2 + m^2} \qquad \text{(for a free particle).}$$
- Negative energy solutions bad: **no stable ground state**! Every state would decay further "down", a unique source of energy.

The Dirac equation

- Dirac realised that this is potentially problematic and that naively spin could not be included with ψ a simple complex number.
- To get rid of the negative energies, he linearised the equation in E $(\partial/\partial t)$ - this was possible only with ψ forced to have at least two components.
- Identify the two components with spin up and down: $\psi = (\psi_\uparrow, \psi_\downarrow)$. Seemingly special relativity enforces spin!
- But how about the **negative energy solutions?** Dirac's suggestion: **hitherto unseen anti-particles!**
- As a result, he finally wrote down an equation with ψ having four components, two for the two spins of positive and two for negative energies.

Anti-particles

- Dirac's proposal for solutions with $E < 0$: Fill the (Dirac-)"sea" of negative energy states, (Fermi-character prevents double fillings and therefore guarantees the stability of the vacuum).
- Can excite them with, e.g., photons.
- Then anti-particles are just "holes" in the sea: **absence of negative energy looks like net positive energy**.
- The related particle (the anti-particle) must have **same mass** as ordinary particles, but **opposite charge**.

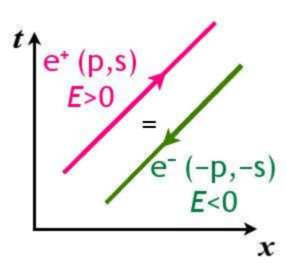


Stueckelberg-Feynman interpretation

- Stueckelberg-Feynman antimatter-interpretation (1947): Negative energy solutions are indeed positive energy solutions of a new particle, moving backwards in time (advanced vs. retarded waves).
- Motivation: Time evolution operator,

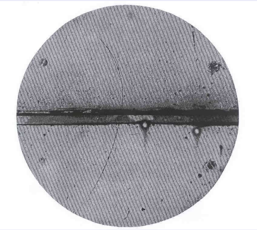
$$U(t, t_0) = \exp[iE(t - t_0)]$$

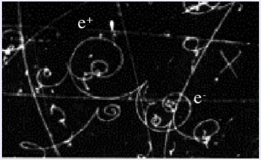
for unperturbed free particle.
- Benefit of this interpretation: treating electrons and positrons on equal footing (no more holes).



Evidence for anti-particles (Andersson, 1932)

- Finding a particle electron's mass but opposite charge: the electron's antiparticle, "positron".
- "On August 2 1932 during the course of photographing cosmicray tracks produced in a vertical Wilson chamber (magnetic field 15,000 gauss) designed in the summer of 1930 by Prof R A Millikan and the writer the track shown in fig 1 was obtained which seemed to be interpretable only on the basis of a particle carrying a positive charge but having the same mass of the same order of magnitude as that normally possessed by a free electron."





Summary

- Reviewed briefly basic idea of perturbation theory.
- Feynman diagrams as terms in the perturbative expansion of the transition amplitude between states.
- Discussed the Dirac equation and antiparticles.
- To read: Coughlan, Dodd & Gripaios, "The ideas of particle physics", Sec 4.