Introduction to particle physics
Lecture 4: Cross sections

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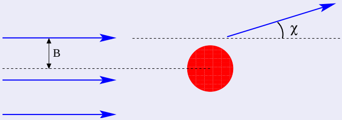
Outline

Cross sections in classical physics

Definition



- Consider a beam of particles, approaching a target at rest with velocity v_∞. Describe the target by a potential centered at the origo.
- Due to different impact parameters B, different particles of the beam are scattered at different angles χ.
- Define cross section $\mathrm{d}\sigma(\chi) = \mathrm{d}N(\chi)/n$ with
  - $\mathrm{d}N(\chi)$ = number of particles scattered per unit time into the interval $[\chi,\, \chi + \mathrm{d}\chi]$ - physical units: s^{-1}.
  - n = number of particles passing per unit time through a unit area perpendiculr to the beam - physical units: $m^{-2}s^{-1}$

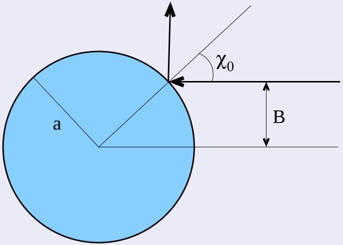
Rewriting the differential cross section $\mathrm{d}\sigma$

- Assume unique relation $\chi = \chi(B)$. (Fulfilled if χ decreases with B increasing - a typical setup!)
- Assume homogenous beams: $n =$ constant.
- Assume symmetry around beam axis.
- Then: $\mathrm{d}N(\chi) = 2\pi n B \mathrm{d}B \Longrightarrow \mathrm{d}\sigma = 2\pi B \mathrm{d}B$.
- Can rewrite to expose dependence on scattering angle:
$$\mathrm{d}\sigma = 2\pi B(\chi) \left|\frac{\mathrm{d}B}{\mathrm{d}\chi}\right| \mathrm{d}\chi.$$
- Use solid angle $\mathrm{d}\Omega = \sin\chi \mathrm{d}\chi \mathrm{d}\phi$:
$$\mathrm{d}\sigma = \frac{B(\chi)}{\sin\chi} \left|\frac{\mathrm{d}B}{\mathrm{d}\chi}\right| \mathrm{d}\Omega.$$

Example: Scattering of a hard sphere with radius a

- Read off from sketch:

$$B = a\sin\chi_0 = a\sin\frac{\pi-\chi}{2} = a\cos\frac{\chi}{2}$$



- Therefore:

$$\mathrm{d}\sigma = 2\pi B\left|\frac{\mathrm{d}B}{\mathrm{d}\chi}\right|\mathrm{d}\chi = 2\pi a^2\cos\frac{\chi}{2}\left|\frac{1}{2}\sin\frac{\chi}{2}\right|\mathrm{d}\chi = \frac{\pi a^2}{2}\sin\chi\mathrm{d}\chi = \frac{a^2}{4}\mathrm{d}\Omega \implies \sigma = a^2\pi\,.$$

- Particle must “hit” the target.

Example: Scattering in a central potential $V = \alpha/r^n$

(Recap the corresponding lecture in classical mechanics! Details of derivation not examinable.)

- Consider now scattering in a central potential $V(r) = \alpha/r^n$.
- Obviously: Need to determine trajectories in dependence on B.
- Use energy and angular momentum conservation:

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} + V(r),$$

(Scattering particles have mass m.)

where two-dimensional spherical coordinates r and ϕ and the angular momentum

$$J = mr^2\dot{\theta}$$

have been used.

- Therefore:

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{2}{m}\left[E - V(r)\right] - \frac{J^2}{m^2r^2}}$$

- Rewrite (with $\mathrm{d}t = \mathrm{d}r/\dot{r}$ from above):

$$\mathrm{d}\theta = \frac{J\mathrm{d}t}{mr^2} = \frac{\frac{J}{r^2}\mathrm{d}r}{\sqrt{2m\left[E - V(r)\right] - \frac{J^2}{r^2}}}\,.$$

- Use that energy in infinite distance is purely kinetic: $E = \frac{mv_\infty^2}{2}$ and angular momentum is given by $J = mv_\infty B$.
- Assume specific form of potential:
  $V(r) = \alpha/r$ (typical for gravity/electromagnetism)

$$\chi_0 = \int \mathrm{d}\theta = \int \frac{\frac{B}{r^2}\mathrm{d}r}{\sqrt{1 - \frac{B^2}{r^2} - \frac{2V(r)}{mv_\infty^2}}} = \cos^{-1}\frac{\frac{\alpha}{mv_\infty^2 B}}{\sqrt{1 + \left(\frac{\alpha}{mv_\infty^2 B}\right)^2}}$$

(Finite terms absorbed in definition of angle/orientation of coordinate system)

- Solve for B and use that $\chi_0 = \frac{\pi-\chi}{2}$:

$$B^2 = \frac{\alpha^2}{m^2 v_\infty^4} \tan^2\chi_0 = \frac{\alpha^2}{m^2 v_\infty^4} \cot^2\frac{\chi}{2}\,.$$

- To express cross section $\mathrm{d}\sigma = 2\pi B \left|\frac{\mathrm{d}B}{\mathrm{d}\chi}\right| \mathrm{d}\chi$

  use that $$\mathrm{d}B/\mathrm{d}\chi = -\frac{\alpha}{2mv_\infty^2}\frac{1}{\sin^2\frac{\chi}{2}}.$$

- Rediscover Rutherford formula

$$\mathrm{d}\sigma = \frac{\pi\alpha^2}{m^2 v_\infty^4}\frac{\cos\frac{\chi}{2}}{\sin^3\frac{\chi}{2}}\mathrm{d}\chi = \frac{\alpha^2}{4m^2 v_\infty^4}\frac{\mathrm{d}\Omega}{\sin^4\frac{\chi}{2}}$$

- Observation: for small angles $\chi \to 0$ differential cross section infinite
  total cross section diverges for Rutherford scattering
- This is a consequence of the infinite range of the potential!

(Will need to “cut” for minimal scattering angle . . .)

Event rates, luminosity and cross sections

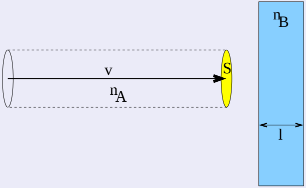
- Introduce (instantaneous) luminosity $\mathcal{L}$,
  the number of incident particles per unit area per unit time times the opacity of the target; units $[\mathcal{L}] = m^{-2}s^{-1}$
- Then the **instantaneous** number of events $\mathrm{d}N_{\mathrm{ev}}/\mathrm{d}t$ reads

$$\frac{\mathrm{d}N_{\mathrm{ev}}}{\mathrm{d}t} = \sigma\mathcal{L}$$

- Therefore the total number of events during a time T given by the integral over time of $\mathcal{L}$, the **integrated luminosity** – cross section σ depends on the particles and the reaction type only.
- Typically a year of beam in an accelerator experiment: $1\,\mathrm{yr} = 10^7$ s.

Determining luminosity

- Consider a simple case: scattering off a fixed target.
  Beam consisting of particles A, travelling with velocity v, uniformly distributed in area S with constant density n_A along beam axis.



- Therefore: number of particles in the beam per unit length is given by $\mathrm{d}N_A = n_A S \mathrm{d}x$, and the incident flux (number of particles per unit time) on target is

$$\Phi_A = \frac{\mathrm{d}N_A}{\mathrm{d}t} = n_A S \frac{\mathrm{d}x}{\mathrm{d}t} = n_A S v$$

- Assume thin (only one scatter per particle) target with size l :
  Area density of particles in target able to participate in reaction

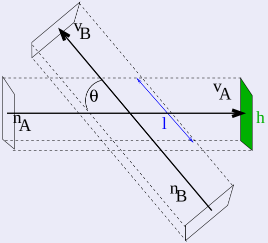
$$\rho_B = n_B l$$

- Define **luminosity**

$$\mathcal{L} = \Phi_A \rho_B = n_A n_B S v l.$$

Luminosity in collider experiments

- Consider now two quadratic beams each with area h^2, colliding under an angle θ, with densities n_A and n_B along their beam axes, and velocities $v_A \geq v_B$.
- Look at beam B as "target", and beam A remaining the "beam".



- Need only to evaluate N_B – can use flux Φ_B:

$$n_B = \frac{\Phi_B}{h^2 v_B}$$

- Therefore density per area

$$\rho_B = n_B \frac{h}{\sin\theta} = \frac{\Phi_B}{h v_B \sin\theta}$$

and

$$\mathcal{L} = \Phi_A \rho_B = \frac{\Phi_A \Phi_B}{\sin\theta\, h v_B}.$$

Practicalities

- Hard to determine luminosity - geometry dependent and fluctuations with quality/homogeinity of the beams
- Therefore: Measure them with "standard candles", i.e. well understood processes with good theoretical accuracy in calculation of cross section σ.

Units

- Recap event rate for events of type $AB \to X$:

$$\dot{N}_{AB\to X} = \frac{\mathrm{d}N_{AB\to X}}{\mathrm{d}t} = \mathcal{L}_{AB}\sigma_{AB\to X}$$

- Units: $[\dot{N}_{AB\to X}] = \text{Hz} = 1/\text{s}, \Longrightarrow [\mathcal{L}] = 1/([\sigma_{AB\to X}]\cdot\, \text{s})$.

$$\begin{aligned}[\sigma] &= 1\ \text{barn} = (10\ \text{fm})^2 = 10^{-24}\ \text{cm}^2 \\ &\approx (2\ \text{GeV})^{-2} = (2000\ \text{MeV}){-2}\end{aligned}$$

Summary

- Recapitulated cross sections in classical physics.
- Calculated example cross sections for two specific cases: hard shell and Coulomb-potential
- Connection to event rates and luminosity.