November 19, 2007

Fundamental Symmetries — Example Sheet 9

9.1 Show that when $F_{\mu\nu}$ is expressed in terms of component electric and magnetic fields $\underline{E}$ and $\underline{B}$, the dual field strength $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ has $\underline{E}$ and $\underline{B}$ interchanged. Show further that $F_{\mu\nu}F^{\mu\nu} = \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$, so that the only two quadratic Lorentz invariants are $\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\underline{B}^2 - \underline{E}^2)$, and $\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = \underline{E}.\underline{B}$.

Deduce the following results: (i) if $\underline{E}$ and $\underline{B}$ are perpendicular in one frame, they will be perpendicular in all frames (ii) if there exists a frame in which $\underline{E}$ is zero, and there is another frame in which $\underline{B}$ is zero, then both $\underline{E}$ and $\underline{B}$ are zero in all frames.

Show that when $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, and thus such a term added to the Lagrangian makes no contribution to the equations of motion.

Comment on the extension of these results to Yang-Mills theories. Show in particular that $\frac{1}{4}\mathrm{tr}F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\mu \mathrm{tr}(A^\nu\partial^\rho A^\sigma + \frac{2}{3}A^\nu A^\rho A^\sigma)$.

9.2 Consider the Lagrangian of scalar electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi^*\phi)\ ,$$

where $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, i.e. a $U(1)$ gauge theory with one massless vector boson and two scalar particles. How many physical degrees of freedom does the theory have?

Consider the case that the $U(1)$ is spontaneously broken:

$$V(\phi^*\phi) = \frac{\lambda}{6}\left(\phi^*\phi - \tfrac{1}{2}c^2\right)^2\ ,\ c = \sqrt{2}\langle 0|\phi|0\rangle \neq 0\ .$$

Derive the new form of the Lagrangian in polar coordinates, i.e. $\phi = \frac{1}{\sqrt{2}}(c+\rho)\,\mathbf{e}^{i\theta}$ $(\theta, \rho$ real). What would be the form of the Langrangian and the role of θ if the gauge field was absent?

Now fix the gauge by defining a field $B_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$ and derive the form of the Lagrangian in terms of the field B_μ. Interpret the role of the θ degree of freedom and find the physical states. Check that the number of degrees of freedom is the same before and after symmetry breaking. List the masses and the interactions between the physical fields together with their couplings.