November 12, 2007

**Fundamental Symmetries — Example Sheet 8**

8.1 Describe the effect of spontaneous symmetry breakdown on the $O(N)$ symmetric real scalar field theory with Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}(\phi^2 - v^2)^2$. Find the number of Goldstone bosons and the residual symmetry group, and thus verify Goldstone's theorem. Perform the same exercise on the $U(N)$ symmetric complex scalar field theory with Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu \phi^\dagger \partial^\mu \phi - \frac{\lambda}{12}(\phi^\dagger\phi - v^2)^2$.

8.2 A theory with eight real scalar fields $\phi^a$ which is $SU(3)$ symmetric may be constructed by writing $\Phi = \lambda^a \phi^a$, where $\lambda^a$ are the Gell-Mann matrices: the Lagrangian $\mathcal{L} = \frac{1}{2}\mathrm{tr}\partial_\mu \Phi^2 - \mathrm{tr}V(\Phi)$ is then invariant under $\Phi \to u\Phi u^\dagger$, $u \in SU(3)$. Show that if two fields $\Phi_1^0$ and $\Phi_2^0$ are both possible vacuum states (and thus related under $SU(3)$), they must both have the same eigenvalues. Discuss the possible residual symmetries after spontaneous breaking (there are two which are nontrivial) and deduce the corresponding numbers of Goldstone bosons.

Discuss the generalization to $SU(N)$, and show that the number of Goldstone bosons is $N^2 - \sum_{i=1}^{r} n_i^2$, where $r$ is some integer between zero and $N$, and $n_i$ are $r$ positive integers which sum to $N$.

Which is the smallest $SU(N)$ which can break in this way to $SU(3) \otimes SU(2) \otimes U(1)$?

8.3 Add a soft breaking term $-\epsilon\sigma$ to the potential for the scalar fields in the linear $SU(2) \otimes SU(2)$ chiral sigma model. Show that the vacuum state is now uniquely defined, and that the Goldstone bosons now acquire a mass $m_\pi^2 = \epsilon/f$ if the soft breaking is small. Show that the vector (isospin) current is still conserved, but that $\partial_\mu \underline{A}^\mu = -m_\pi^2 f \underline{\pi}$ (the 'partially conserved axial current').

By comparing this result to the similar expression for soft breaking of the conservation of the axial current $\partial_\mu \underline{A}^\mu = -2m\bar{\psi}\underline{\tau}\gamma^5\psi$ in the quark model, deduce that $m_\pi^2$ must be proportional to the quark mass $m$. Deduce a Gell-Mann-Okubo mass formula for pseudoscalar mesons.

8.4 In the quark model the chiral charges are $Q_a = -i\int d^3\underline{x}\psi^\dagger T_a \psi$ and $Q_a^5 = -i\int d^3\underline{x}\psi^\dagger\gamma^5 T_a\psi$, where the $T_a$ are (hermitian) generators of $SU(N)$. Show that these charges satisfy an $SU(N) \otimes SU(N)$ chiral algebra if the quark fields satisfy *anti*-commutation relations

$$\begin{aligned} \{\psi_r(\underline{x},t), \psi_s(\underline{y},t)\} &= \{\psi_r^\dagger(\underline{x},t), \psi_s^\dagger(\underline{y},t)\} = 0, \\ \{\psi_r(\underline{x},t), \psi_s^\dagger(\underline{y},t)\} &= i\delta^3(\underline{x}-\underline{y})\delta_{rs}, \end{aligned}$$

where $r, s$ are flavour indices or spinor indices.

[Hint: first show that $[AB, CD] = A\{B,C\}D - \{A,C\}BD + CA\{B,D\} - C\{A,D\}B$.]