Fundamental Symmetries — Example Sheet 7

7.1 Show that the kinetic term for N real scalar fields ϕ_i is invariant if the fields transform according to the fundamental representation of $O(N)$. Show that for N complex scalar fields, or Weyl spinors, the corresponding invariance is under the fundamental representation of $U(N)$, while for Dirac spinors it is $U(N) \otimes U(N)$.

Show that if T^a are the generators of $O(N)$, the bilinears $\phi^T T^a \phi$ transform according to the adjoint representation. Construct similar bilinears for complex scalars, Weyl and Dirac spinors.

7.2 Show that for a scalar field theory with an internal symmetry represented infinitessimally as $\delta\phi_r = -i\alpha^a T^a_{rs}\phi_s$, where the matrices T^a generate the Lie algebra of a compact group G, i.e. $[T^a, T^b] = ic^{abc}T^c$, the charges $Q^a = -i\int d^3\underline{x}\,\pi_r T^a_{rs}\phi_s$ are conserved, where $\pi_r \equiv \partial\mathcal{L}/\partial\dot{\phi}_r$. Show further that if $\phi_r(x)$ and $\pi_r(x)$ satisfy the equal time commutation relations

$$\begin{aligned}
[\phi_r(\underline{x}, t), \phi_s(\underline{y}, t)] &= [\pi_r(\underline{x}, t), \pi_s(\underline{y}, t)] = 0, \\
[\phi_r(\underline{x}, t), \pi_s(\underline{y}, t)] &= i\delta^3(\underline{x} - \underline{y})\delta_{rs},
\end{aligned}$$

then the charges Q^a also generate the Lie algebra of G, i.e.

$$[Q^a, Q^b] = ic^{abc}Q^c, \qquad [Q^a, \phi_r] = -T^a_{rs}\phi_s.$$

7.3 In the $SU(3)$ quark model there are two singlet vector states $|\omega_8\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle)$ belonging to the octet and the 'pure' singlet $|\omega_1\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle)$ (cf. Q6.4). The two physical states are the ω and ϕ mesons. The ω decays primarily into three pions, while the ϕ prefers to decay into kaons: show that this suggests that the the mass eigenstates $|\omega\rangle$ and $|\phi\rangle$ are mixtures of $|\omega_8\rangle$ and $|\omega_1\rangle$ with a mixing angle $\tan\theta \sim \sqrt{2}$. Apply the Gell-Mann–Okubo formula to deduce that $m_{K^*} = \frac{1}{4}(m_\rho + m_\omega + 2m_\phi)$. How well does this work in practice? ($m_{K^*} = 892\,\mathrm{MeV}$)

7.4 Use arguments similar to those used to derive the Gell-Mann–Okubo formula to show that

$$d_{3ab}\lambda_a\lambda_b = 2(V^2 - U^2) + I_3 Y, \qquad \tfrac{1}{\sqrt{3}}d_{8ab}\lambda_a\lambda_b = \tfrac{2}{3}(2I^2 - U^2 - V^2) + \tfrac{1}{3}I_3^2 - \tfrac{1}{4}Y^2,$$

where I^2, U^2 and V^2 are the operators charachterising the total I-spin, U-spin and V-spin, while I_3 and Y are the isospin and hypercharge.

Since in $SU(3)$ the electric charge $Q = I_3 + \frac{1}{2}Y$, it seems reasonable to assume that the magnetic moments of hyperons are given by matrix elements of a magnetic moment operator $\mu = \mu_3 + \frac{1}{\sqrt{3}}\mu_8$, with μ_3 and μ_8 components of an octet of operators μ_a. By writing $\mu_a = \alpha\lambda_a + \beta d_{abc}\lambda_b\lambda_c$, with α and β unknown constants, show that

$$\mu = \alpha[2I_3 + Y] + \beta[\tfrac{4}{3}(I^2 + V^2 - 2U^2) + \tfrac{1}{3}I_3^2 - \tfrac{1}{4}Y^2 + YI_3].$$

Use this result to deduce the predictions

$$\mu_{\Sigma^+} = \mu_p, \qquad \mu_{\Xi^0} = \mu_n \qquad \mu_{\Xi^-} = \mu_{\Sigma^-} = -\mu_p - \mu_n.$$