October 29, 2007

Fundamental Symmetries — Example Sheet 6

6.1 (Revision) Use the result of Q2.6 to construct explicitly the decompositions $\mathbf{2}\otimes\mathbf{2} = \mathbf{3}\oplus\mathbf{1}$, $\mathbf{3}\otimes\mathbf{2} = \mathbf{4}\oplus\mathbf{2}$, and $\mathbf{3}\otimes\mathbf{3} = \mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}$, where $\mathbf{N}$ is the representation of $su(2)$ with $\mathbf{N}$ states, and thus spin $j = \frac{1}{2}(N-1)$.

6.2 Compute the structure constants f_{ijk} for $su(3)$ by commuting the Gell-Mann matrices $\{\lambda_i\}$ (remembering to exploit the antisymmetry). Show that with this normalization $g_{ij} = 4f_{ikl}f_{jkl} = 12\delta_{ij}$.

Show by expanding the anticommutator in invariant tensors that

$$\{\lambda_i, \lambda_j\} = \tfrac{4}{3}\delta_{ij} + 2d_{ijk}\lambda_k.$$

Use this result to compute the nonzero components of the symmetric invariant tensor $d_{ijk} \equiv \frac{1}{4}\mathrm{tr}\{\lambda_i, \lambda_j\}\lambda_k$.

6.3 Show that for representations of $su(3)$:

(i) There is at most one singlet state if the representation is irreducible;

(ii) If $N(p,q)$ is the number of states in a representation $\mathbf{N}$ specified by (p,q), then (a) $N(p,q) = N(q,p)$, (b) for triangular representations $N(p,0) = \frac{1}{2}(p+1)(p+2)$ and finally (c) $N(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2)$ [assuming the result is true for $N(p-1,q-1)$ show that it is true for $N(p,q)$, use (b) to start the induction];

(iii) Show that $\mathbf{N} = \mathbf{N}^*$ (i.e. the representation is real) when N is the cube of an integer, and thus that the representations $\mathbf{1}, \mathbf{8}, \mathbf{27}, \ldots$ are all real.

6.4 Label the states in the $\mathbf{3}$ representation of $su(3)$ by $|u\rangle$, $|d\rangle$, $|s\rangle$ ('quarks'), corresponding to $I_3 = \frac{1}{2}, -\frac{1}{2}, 0$ respectively. Show using the commutation relations that $V_+U_-I_-|u\rangle = |u\rangle$ and thus that the phase conventions

$$I_-|u\rangle = |d\rangle, \qquad U_-|d\rangle = |s\rangle, \qquad V_+|s\rangle = |u\rangle,$$

are consistent. Use these results to generate all the states in the reducible representation $\mathbf{3}\otimes\mathbf{3}$, decomposing it as $\mathbf{6}\oplus\mathbf{3}^*$. [Take $|u\rangle|u\rangle$ as the state of greatest weight, and lower it with I_- and V_-, then find the state of greatest weight in the $\mathbf{3}^*$ by orthogonalisation.]

Show similarly that if the states in the $\mathbf{3}^*$ representation are labelled by $|\bar{u}\rangle$, $|\bar{d}\rangle$, $|\bar{s}\rangle$ ('antiquarks'), then $V_-U_+I_+|\bar{u}\rangle = -|\bar{u}\rangle$, and thus that a consistent phase convention is

$$I_+|\bar{u}\rangle = -|\bar{d}\rangle, \qquad U_+|\bar{d}\rangle = -|\bar{s}\rangle, \qquad V_-|\bar{s}\rangle = -|\bar{u}\rangle.$$

Now generate all the states in the reducible representation $\mathbf{3}\otimes\mathbf{3}^* = \mathbf{8}\oplus\mathbf{1}$, noting that the isosinglet state is $\frac{1}{\sqrt{3}}(|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle + |s\rangle|\bar{s}\rangle)$ because it is annihilated by all the roots.

Finally compute the decomposition $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{10}\oplus\mathbf{8_S}\oplus\mathbf{8_A}\oplus\mathbf{1}$: here the singlet is totally antisymmetric in the exchange of any two quarks, while the two octets are respectively symmetric and antisymmetric in the exchange of the first two quarks.