Fundamental Symmetries — Example Sheet 5

5.1 Show that if a transformation $\Phi \to \Phi + \alpha \partial\Phi/\partial\alpha$ is *not* a symmetry of the Lagrangian, then the Noether current is no longer conserved, but rather $\partial_\mu J^\mu = \partial\mathcal{L}/\partial\alpha$. Use this result to show that for a massive Dirac fermion the conservation of the chiral current is softly broken by the mass term: $\partial^\mu j^5_\mu = -2m\bar{\psi}\gamma^5\psi$.

5.2 Construct the canonical stress-energy tensor $T_{\mu\nu}$ for a scalar field theory with Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - \tfrac{\lambda}{4!}\varphi^4,$$

and check that it is both symmetric and conserved. Show that by adding to $T_{\mu\nu}$ a term proportional to $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\varphi^2$ it is possible to construct an 'improved' stress-energy tensor $\Theta_{\mu\nu}$ which is conserved, symmetric and traceless when the scalar field is massless.

5.3 Consider the Lagrangian

$$\mathcal{L} = \bar{\psi}\gamma^\mu\partial_\mu\psi + ig\bar{\psi}(\sigma + i\gamma^5\pi)\psi + \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\pi)^2 - V(\sigma^2 + \pi^2),$$

where σ and π are real scalar fields. Find the field equations. Show that the Lagrangian is invariant under the chiral symmetry

$$\delta\psi = i\beta\gamma^5\psi, \qquad \delta\sigma = 2\beta\pi, \qquad \delta\pi = -2\beta\sigma,$$

and calculate the corresponding chiral current J^5_μ. Confirm that on using the field equations $\partial^\mu J^5_\mu = 0$. What is the spectrum of this theory when (a) $V(\sigma^2+\pi^2) = \frac{1}{2}m^2(\sigma^2+\pi^2)$, or (b) $V(\sigma^2 + \pi^2) = \lambda(\sigma^2 + \pi^2 - m^2)^2$?

5.4 Consider a compact semi-simple Lie algebra. Show that the value $c^{(2)}$ of the quadratic Casimir operator $C^{(2)} \equiv \sum_{ab} g_{ab}T_aT_b$ in a hermitian representation is real and positive, and that if h_i are the (real) eigenvalues of the elements H_i of a Cartan subalgebra, then $h_i^2 \leq c^{(2)}$.

Show that for any compact semi-simple Lie algebra with Cartan subalgebra $\{H_i\}$ and step operators $\{E_{\pm\alpha}\}$ that in Cartan-Weyl normalization the quadratic Casimir operator may be written as

$$C^{(2)} = H.H + \sum_{\alpha>0}(\alpha.H + 2E_{-\alpha}E_\alpha).$$

Deduce that for a given representation, if h_i are the eigenvalues of H_i for the state of greatest weight, $c^{(2)} = h.(h + \rho)$, where $\rho_i \equiv \sum_{\alpha>0}\alpha_i$ is called the 'Weyl vector'.

[This result generalizes the $j(j+1)$ result in $su(2)$.]