Fundamental Symmetries — Example Sheet 4

4.1 (Revision) Consider the action of the rotation group $SO(2)$ on the functions $\psi(x_i)$, $x_i \in R^2$: $g\psi(x) = \psi(R^{-1}x)$, with $R^{-1} = R^T$. Show by writing $R = \left(\begin{smallmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{smallmatrix}\right)$ and expanding $\psi(R^{-1}x)$ in powers of ϕ (using the definition of the hermitian generator $T = -i\partial g/\partial\phi|_{\phi=0}$) that (an infinite dimensional) representation of the generators of SO(2) is given by

$$T\psi = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\psi.$$

Hence show that if $x_i \in R^3$, the generators of $SO(3)$ acting on wave functions are represented by the angular momentum operators $L_i = -i\epsilon_{ijk}x_j\partial/\partial x_k$, satisfying the $so(3)$ algebra $[L_i, L_j] = i\epsilon_{ijk}L_k$.

Show similarly that the translation group is generated by momentum operators $P_i = -i\partial/\partial x_i$, and deduce the commutation relations

$$[P_i, P_j] = 0, \qquad [L_i, L_j] = i\epsilon_{ijk}L_k, \qquad [L_i, P_j] = i\epsilon_{ijk}P_k$$

for the algebra of translations and rotations in R_3. Is this algebra semi-simple?

Show by explicit computation that P^2 and $L.P$ are Casimir operators and that L^2 is not. Explain why in a given representation P^2 may take any real value. but $L.P/\sqrt{P^2} = m$, $m = 0, \pm\frac{1}{2}, \pm 1, \ldots$, and thus that helicity is quantised while momentum is not.

4.2 Consider the subgroup of the Lorentz group which leaves the momentum vector p_μ invariant, i.e. all $\Lambda_\mu{}^\nu$ such that $\Lambda_\mu{}^\nu p_\nu = p_\mu$ (this is called 'the little group'). Show that an infinitessimal transformation in the little group may be written in the form $U(\Lambda) = 1 + in_\mu W^\mu$, where $W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}M^{\nu\sigma}P^\rho$.

Explain why W_μ is a vector under Lorentz transformations, and thus write down its commutation relations with the Poincaré generators P_μ and $M_{\mu\nu}$. Use these to show that

$$[W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma.$$

Deduce that if p_μ is timelike, so $p^2 = m^2 > 0$, we can by choosing a suitable frame take $W_0 = 0$ while $S_i \equiv \frac{1}{m}W_i$ satisfies the $so(3)$ algebra, and thus that $W^2 = m^2 s(s+1)$, $s = 0, \frac{1}{2}, 1, \ldots$.

Show similarly that if p_μ is spacelike ($p^2 < 0$), the algebra of the little group is $so(2,1)$, while if p_μ is lightlike ($p^2 = 0$) it is the algebra of translations and rotations in two dimensions.

4.3 The light-cone, defined as all dx_μ such that $\eta_{\mu\nu}dx^\mu dx^\nu = 0$, is invariant under space-time translations and rotations, and scale changes (or 'dilations') $x_\mu \to \alpha x_\mu$. Show that it is also invariant under the discrete transformation $x_\mu \to -x_\mu/x^2$. By combining two of these discrete transformations with a translation $x_\mu \to x_\mu + c_\mu$, show that it is invariant under the 'conformal' transformations

$$x_\mu \to \frac{x_\mu - x^2 c_\mu}{1 - 2x_\mu c^\mu + c^2 x^2}.$$

Find the infinitessimal form of dilations and conformal transformations, and deduce that they may be generated by

$$D \equiv i x^\mu \partial_\mu, \qquad K_\mu \equiv i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu).$$

The set of operators $\{P_\mu, L_{\mu\nu}, D, K_\mu\}$ together generate the fifteen dimensional conformal group. By writing $P_\mu = i\partial_\mu$, $L_{\mu\nu} = i(x_\nu \partial_\mu - x_\mu \partial_\nu)$, show that besides the usual commutation relations of the Poincaré algebra we now have in addition

$$\begin{aligned}
[P_\mu, D] &= iP_\mu; & [P_\mu, K_\nu] &= 2i(\eta_{\mu\nu}D + L_{\mu\nu}); \\
[L_{\mu\nu}, D] &= 0; & [L_{\mu\nu}, K_\lambda] &= i(\eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu); \\
[K_\mu, K_\nu] &= 0; & [D, K_\mu] &= iK_\mu.
\end{aligned}$$

Show that P^2 is no longer a Casimir operator, and thus that only massless theories can be conformally invariant.

By setting $M_{\mu\nu} = L_{\mu\nu}$, $M_{\mu 5} = \frac{1}{2}(P_\mu + K_\mu)$, $M_{\mu 6} = \frac{1}{2}(P_\mu - K_\mu)$, $M_{56} = -D$ show that the conformal algebra is isomorphic to $so(4,2)$, and is thus semi-simple, with three Casimir operators. The covering group is actually $SU(2,2)$.