Fundamental Symmetries — Example Sheet 3

3.1 Consider the basis

$$t_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad t_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for $sl(2,R)$. Write down the commutation relations, deduce generators in the adjoint representation, and compute the Killing form. Is $sl(2,R)$ semi-simple? Is it simple? Is it compact?

Explain why in the defining representation the generator of so(2) (rotations) may be chosen as t_2, while the generator of $so(1,1)$ (boosts) may be chosen as t_1. Write down three generators of $so(2,1)$ in the defining representation (i.e. one rotation and two boosts), and show that these are in one-to-one correspondence with the adjoint representation generators of $sl(2,R)$, so that $so(2,1) \cong sl(2,R)$.

3.2 Show that a matrix corresponding to an infinitessimal rotation in R^N has the form $R_{ij} = \delta_{ij} + \omega_{ij}$, where $\omega_{ij} = -\omega_{ji}$. Hence show that the $\frac{1}{2}N(N-1)$ hermitian $N \times N$ matrices $(M_{ij})_{kl} = i(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$ may represent the generators of $O(N)$, since $R = 1 + \frac{i}{2}\omega_{ij}M_{ij}$. Deduce the commutation relations

$$[M_{ij}, M_{kl}] = i(\delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{jk}M_{il}).$$

Show further that when $N = 4$, if we define $J_1 = M_{23}$, $J_2 = M_{31}$, $J_3 = M_{12}$, $K_i = M_{0i}$, $i = 1, 2, 3$, and $N_i^{\pm} = J_i \pm K_i$, then N_i^+ and N_i^- each satisfy $so(3)$ commutation relations. Deduce that $so(4) \cong so(3) \oplus so(3)$.

Show also that $so(3,1) \cong sl(2,C)$.

3.3 For every $x \in R^4$ we can construct an hermitian 2×2 matrix

$$H \equiv x_0 + x_j\sigma_j = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix},$$

Show that the length $\eta^{\mu\nu}x_\mu x_\nu \equiv x_0^2 - x_1^2 - x_2^2 - x_3^2$ is just $\det H$, that thus that the map $H \to H' = KHK^\dagger$, with $K \in SL(2,C)$ induces a linear map $x_\mu \to x'_\mu = \Lambda_\mu{}^\nu(K)x^\nu$ which preserves $\eta^{\mu\nu}x_\mu x_\nu$. Show further that this mapping is two-to-one, and thus that $SO(3,1) \cong SL(2,C)/Z_2$. Explain why there is an $SU(2) \subset SL(2,C)$ which covers the $SO(3) \subset SO(3,1)$.

[This problem shows that $SL(2,C)$ is the covering group for the Lorentz group, just as $SU(2)$ is the covering group for the rotation group. Similar arguments may be used to show that $SO(4) \cong SU(2) \otimes SU(2)/Z_2$, while $SO(2,2) \cong SL(2,R) \otimes SL(2,R)/Z_2$.]

3.4 Show that for a LH Weyl spinor the generator of Lorentz transformations $S_{\mu\nu} = \frac{i}{4}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu)$, while for a RH spinor $S_{\mu\nu} = \frac{i}{4}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu)$. Deduce that for a Dirac spinor $S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$.

3.5 Show that if the matrices γ_μ satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = \eta^{\mu\nu}$, so do the matrices $U^\dagger\gamma_\mu U$ provided that U is unitary.

Show that the matrix $U = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ transforms the Weyl representation into the Dirac representation $\gamma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\gamma_i = \left(\begin{smallmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{smallmatrix}\right)$. Show further that in the Dirac representation $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, while S_{ij} is the same as in the Weyl representation.

3.6 (i) Confirm that if γ_μ are *any* matrices satisfying the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta^{\mu\nu}$, for some real metric tensor $\eta_{\mu\nu}$, then $S_{\mu\nu} \equiv \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ satisfies the algebra

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(\eta_{\mu\rho}S_{\nu\sigma} + \eta_{\nu\sigma}S_{\mu\rho} - \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\nu\rho}S_{\mu\sigma}).$$

[Hint: use the fact that $\gamma_\mu\gamma_\nu = \frac{1}{2}([\gamma_\mu\gamma_\nu] + \{\gamma_\mu, \gamma_\nu\})$.]

(ii) Show that if $\gamma_\mu^\dagger = P\gamma_\mu P$ for some P such that $P^2 = 1$, then if $\Lambda \equiv \exp(\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu})$ is a Lorentz transformation, $\Lambda^\dagger = P\Lambda^{-1}P$. Deduce that if under this transformation $\psi \to \Lambda\psi$,

$$\bar{\psi} \equiv \psi^\dagger P \to \bar{\psi}\Lambda^{-1},$$

and thus that $\bar{\psi}\psi$ is a Lorentz scalar.

(iii) Show that $[\gamma_\mu, \omega^{\rho\sigma}S_{\rho\sigma}] = i\omega_{\mu\nu}\gamma^\nu$, and thus (by exponentiation) that

$$\Lambda^{-1}\gamma_\mu\Lambda = \Lambda_\mu^{\ \nu}\gamma_\nu,$$

where $\Lambda_\mu^{\ \nu} = \delta_\mu^{\ \nu} - \omega_\mu^{\ \nu} + \ldots$. Deduce that $\bar{\psi}\gamma_\mu\psi$ is a Lorentz vector.

[For Euclidean spaces we may take $P = 1$, while for spaces with metric $(+ - - - \ldots)$ we generally choose $P = \gamma_0$, so that for spatial indices $\gamma_i^\dagger = -\gamma_i$, and $\gamma_i^\dagger\gamma_i = -\gamma_i^2 = 1$.]