Fundamental Symmetries — Example Sheet 2

2.1 Show explicitly from the definition of the Killing form and the Jacobi identity for the structure constants that $c_{abc} \equiv g_{ad} c^d_{bc} = c_{bca} = c_{cab}$, and is thus totally antisymmetric in all three indices.

Why are there no nonabelian Lie groups with dimension two?

2.2 Suppose that $(T_a)_{ij}$ form a representation of a Lie algebra, and that a_i, $a_i^\dagger$, are annihilation and creation operators satisfying the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0.$$

Show that the operators $\chi_a \equiv a_i^\dagger (T_a)_{ij} a_j$ also represent the Lie algebra.

2.3 Consider a semi-simple algebra. Show explicitly using the commutation relations for the generators that if we define a quadratic Casimir operator $C_2 \equiv g^{ab} T_a T_b$, then $[C_2, T_a] = 0$.

2.4 If T_a are the generators of a semi-simple Lie algebra, show that $\mathrm{tr} T_a = 0$.

2.5 Consider the symmetric invariant tensors $d_{a_1,\ldots,a_n} \equiv \mathrm{str} T_{a_1} \ldots T_{a_n}$, where the 'symmetrized trace' str is defined as the trace averaged over all permutations of the matrices inside the trace.

(a) Show that for the algebras $so(N)$, $d_{a_1,\ldots,a_n} = 0$ whenever n is odd.

(b) Show that the same is true for $sp(2N)$ [Hint: use the results of Q1.5, and then consider ETE^T, where $T \in sp(2N)$ and $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.]

(c) Assuming that the number of independent symmetric invariant tensors is no greater than the trace of unity in the defining representation, explain why the rank r of $su(N)$ is $N-1$, while $so(2N)$, $so(2N+1)$ and $sp(2N)$ all have rank N.

(d) Show using the results of Q1.1 and Q1.5 for the dimensions d that $d - r$ is in fact even for all these groups.

(e) Can the algebras $so(6)$, $su(4)$, and $so(3) \oplus so(3) \oplus so(3) \oplus so(3) \oplus so(3)$ be isomorphic?

2.6 (Revision) Show that if $\{T_i\}$ are generators of $su(2)$, so $[T_1, T_2] = iT_3$ etc., then if $T_\pm = T_1 \pm iT_2$, $T^2 = T_1^2 + T_2^2 + T_3^2$, then $T_\pm T_\mp = T^2 - T_3(T_3 \mp 1)$. Deduce that for normalized eigenstates $|j, m\rangle$, where $T^2|j,m\rangle = j(j+1)|j,m\rangle$, $T_3|j,m\rangle = m|j,m\rangle$,

$$T_\pm |j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j, m \pm 1\rangle.$$

Explain why this means that j must be integer or half-integer, and why the number of states in the irreducible representation labelled by j is $2j+1$.