Fundamental Symmetries — Example Sheet 1

1.1 Show that the groups listed below have the following dimensions ($N \geq 2$):

$GL(N,R)$	$GL(N,C)$	$SL(N,R)$	$SL(N,C)$	$O(N)$	$SO(N)$	$U(N)$	$SU(N)$
N^2	$2N^2$	N^2-1	$2(N^2$-1)	$\frac{1}{2}N(N$-1)	$\frac{1}{2}N(N$-1)	N^2	N^2-1

1.2 Explain why the Kronecker delta δ_{ij} is an invariant tensor under orthogonal group transformations. Show also that the totally antisymmetric symbol $\epsilon_{ij\ldots n}$ is invariant under special orthogonal group transformations. Consider the extension of these results to pseudo groups, and use them to write down two invariants made from a length element dx_μ.

1.3 (a) Explain why $gl(N,R) \cong u(1) \oplus sl(N,R)$, $gl(N,C) \cong u(1) \oplus u(1) \oplus sl(N,C)$, $u(N) \cong u(1) \oplus su(N)$ and $o(N) \cong so(N)$.
(b) Use the results of ex.(1.1) to list all the special groups considered there with dimension $d = 1$, with $d = 2$, etc. up to $d = 10$. Add product groups such as $SU(2) \otimes SU(2)$ to your list. Suggest which of all these groups may have the same Lie algebras.

1.4 (a) Show by considering an explicit choice of generators that $so(3) \cong su(2)$.
(b) Consider the transformation $M \to M' = UMU^{-1}$, where M is a 2×2 traceless hermitian matrix and $U \in SU(2)$. Show by writing $M = x_i\sigma_i$, where σ_i are the Pauli matices, that for every U there is a matrix $R_{ij} \in SO(3)$ such that $x_i \to x'_i = R_{ij}x_j$. Show further that this mapping is two-to-one, and thus that $SO(3) \cong SU(2)/Z_2$.

1.5 The real symplectic group $Sp(2N)$ may be defined as elements of $GL(2N,R)$ which leave $x_i\tilde{y}_i - \tilde{x}_i y_i$ invariant for all vectors $(x_1,\ldots,x_N,\tilde{x}_1,\ldots,\tilde{x}_N), (y_1,\ldots,y_N,\tilde{y}_1,\ldots,\tilde{y}_N) \in R^{2N}$. Show that if the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2N)$ then

$$A^TC = C^TA, \qquad B^TD = D^TB, \qquad A^TD - C^TB = 1,$$

and thus that $Sp(2N)$ has dimension $2N^2 + N$.

Find related conditions for the generators. Explain why for $N = 1$ $Sp(2) \cong SL(2,R)$ while for $N > 1$ $sp(2N)$ is a proper subalgebra of $sl(2N,R)$.