# Introduction to Quantum Field Theory

Rodrigo Alonso
typeset and artwork by Mia West

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## Contents

# Prelude

## 0.1 References

For further reading and reference:

- Quantum Field Theory and the Standard Model
  Matthew D. Schwartz
- An Introduction to Quantum Field Theory
  Michael E. Peskin and Daniel V. Schroeder
- Quantum Field Theory in a Nutshell
  Anthony Zee
- Quantum Field Theory Lectures
  David Tong
  www.damtp.cam.ac.uk/user/tong/qft.html
- Sidney Coleman's QFT Lectures
  https://arxiv.org/abs/1110.5013
  www.physics.harvard.edu/events/videos/Phys253

## 0.2 Conventions

These lecture notes use the following conventions.

### 0.2.1 Vectors

$$x^\mu = (ct, \vec{x}) \qquad\qquad p^\mu = \left(\frac{E_p}{c}, \vec{p}\right)$$

where $c$ is the speed of light. Greek letters will be used for spacetime indices $\mu = 0, 1, 2, 3$ that can be understood to correspond to $t, x, y, z$.

We use the Minkowski metric in the mostly-minus convention

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

which obeys $\eta^{\mu\rho}\eta_{\rho\nu} \equiv \delta^\mu_\nu$, such that

$$\begin{aligned} x_\mu &= \eta_{\mu\nu} x^\nu = (ct, -\vec{x}) \\ \partial_\mu &= \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial (ct)}, \vec{\nabla}\right) \end{aligned}$$

We can also write the quantum mechanical energy and momentum operators into the combined operator $p^\mu$ as

$$p^\mu = \left(\frac{E}{c} = \frac{i\hbar}{c}\frac{\partial}{\partial t} \quad , \quad \vec{p} = -i\hbar\vec{\nabla}\right) = i\hbar\partial^\mu$$

### 0.2.2 Dimensional Analysis

Eventually, we will use **natural units** which is standard in quantum field theory

$$\begin{aligned} c &= 2.988 \times 10^8 \text{ ms}^{-1} = 1 \\ \hbar &= 1.055 \times 10^{-34} \text{ Js} = 1 \end{aligned}$$

which gives all quantities dimensions of energy (usually in units of electronvolt, $eV$) to some power.

We usually denote the mass dimension of a quantity within square brackets, such as

$$[\text{mass}] = 1 \qquad\qquad [\text{length}] = -1$$

and some more relevant quantities:

$$\begin{aligned} [E] = [\partial_\mu] = [p_\mu] &= 1 \\ [dx] = [x] = [t] &= -1 \end{aligned}$$

which you can derive by thinking about how physical quantities relate to each other with equations from quantum mechanics and special relativity, for example:

$$\begin{aligned} E^2 &= m^2c^4 + p^2c^2 \\ p &= \hbar k \\ x^0 &= ct \end{aligned}$$

where $k$ here is a wavenumber (i.e. the number of waves per unit length).

## 0.3 Useful Formulae

Here are some formulae you might find helpful for this course.

- **The Dirac Delta**

$$\int_{-\infty}^{\infty} dk\, e^{ikx} = (2\pi)\delta(x) \tag{1}$$

- **Commutators**

$$[A, BC] = [A, B]C + B[A, C] \tag{2}$$

- **Cauchy's Integral Formula**
  **For $f(z)$ analytic within the curve $C$, such as in fig. 1**

$$\oint dz \frac{f(z)}{z - z_0} = 2\pi i f(z_0) \tag{3}$$

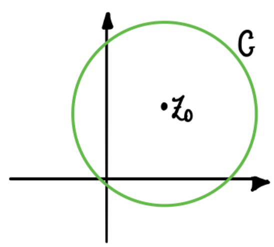



Figure 1: Example contour C on the complex plane

# 1 Quantum Field Theory: Why, What and How?

The language that our most fundamental description of Nature is written on is quantum field theory. Learning this language takes time and (well spent) energy, so it is good to first pause and see what is in it for us by answering three questions.

## 1.1 Why do we need quantum field theory?

We do because:

a) We need a consistent quantum treatment of relativistic particles.

b) Our theory should be local & causal, as nature has shown itself to be.

c) Our theory should allow for a dynamical number of particles.

Let's elaborate on what we mean by each of this points, while the way in which QFT addresses them will be unvelied throughout the course.

### 1.1.1 a) Consistent Relativistic Quantum Theory

Take **non-relativistic quantum mechanics**. I'm guessing you already know about:

- Quantum mechanical states: $|\psi\rangle$
- Wavefunctions: $\langle x|\psi\rangle = \psi(x)$
- The Schroedinger equation: $i\hbar\frac{\partial}{\partial t}\psi(x) = -\frac{\hbar^2\nabla^2}{2m}\psi(x)$
- The energy of a particle with momentum $\vec{p}$: $E_p = \frac{|\vec{p}|^2}{2m}$

Now try the same thing for **relativistic quantum mechanics**. One has $E_p = \sqrt{m^2c^4 + \vec{p}^2c^2}$ and we can try to plug this into the Schroedinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(x) = \text{“}\sqrt{m^2c^4 - \nabla^2\hbar^2c^2}\text{''}\,\psi(x) \tag{4}$$

But we run into a problem: eq. (4) isn't a valid parital differential equation (because of this $\sqrt{\nabla^2}$), so how about we try squaring it

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi = \left(m^2c^4 - \nabla^2\hbar^2c^2\right)\psi \tag{5}$$

which has a solution

$$e^{-iE_pt/\hbar + i\vec{p}\cdot\vec{x}/\hbar} \tag{6}$$

and $E_p^2 = m^2c^4 + \vec{p}^2c^2$. Unfortunately, when we take the square root, there are two solutions:

$$E_p = \pm\sqrt{m^2c^4 + \vec{p}^2c^2} \tag{7}$$

one of which has negative energy. Bad news - negative energy states are associated with unstable systems: a continuum of negative energy until $-\infty$!

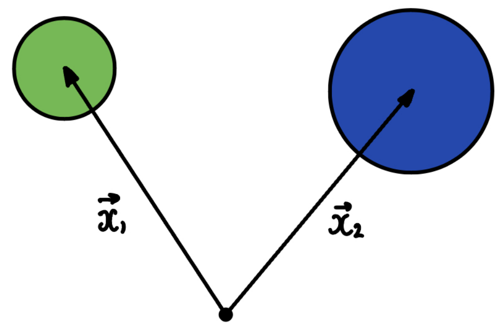



Figure 2: Massive, charged objects $m_1, Q_1$ and $m_2, Q_2$ at positions $x_1$ and $x_2$.

#### 1.1.2 b) Locality & Causality

Take two massive, charged objects $m_1, Q_1$ and $m_2, Q_2$ at positions $x_1$ and $x_2$ respectively, sketched in fig. 2. The potential energy they feel classically is given by Newton's law of gravitation & Coulomb's law:

$$V = -G_N \frac{m_1 m_2}{|\vec{x}_1 - \vec{x}_2|} + \frac{Q_1 Q_2}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_2|} \tag{8}$$

But does object 1 know of object 2 instantly? Clearly not, since for example we know if the sun disappeared, it would take us $\sim 8$ minutes for us to realise.

The solution to this is that fields mediate the interaction and travel at a speed $v \leq c$. Recall from classical electromagnetism, Maxwell's equations in terms of a scalar potential $\Phi = \Phi(\vec{x}, t)$ and vector potential $\vec{A} = \vec{A}(\vec{x}, t)$:

$$-\nabla^2\Phi - \partial_t \vec{\nabla} \cdot \vec{A} = \rho/\epsilon_0 \tag{9}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} + \frac{1}{c^2}\frac{2}{\epsilon}\left(\vec{\nabla}\Phi + \partial_t \vec{A}\right) = \mu_0 \vec{J} \tag{10}$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \tag{11}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{12}$$

such that the potential felt by object 1 is sourced by object 2, which we can write as follows:

$$\Phi(\vec{x}, t) = \frac{Q_2 c}{\epsilon_0} \int_\infty^t \frac{dt' d^3k}{(2\pi)^3 2|\vec{k}|} e^{i\vec{k}(\vec{x} - \vec{x}_2(t'))} \left(ie^{-i|\vec{k}|(t-t')c} + h.c.\right) \tag{13}$$

where the exponential terms describe waves travelling at the speed of light. Now we are able to formulate the dynamics of object 1 locally, for example the potential energy of object 1 as a result of 2 is:

$$V(\vec{x}_1) = Q_1 \Phi(\vec{x}_1, t) \tag{14}$$

#### 1.1.3 c) Variable Number of Particles

In quantum mechanics, the number of e.g. electrons is fixed and does not change. The combination of special relativity & quantum mechanics does, however, imply that particle number is a dynamical (changing) quantity.

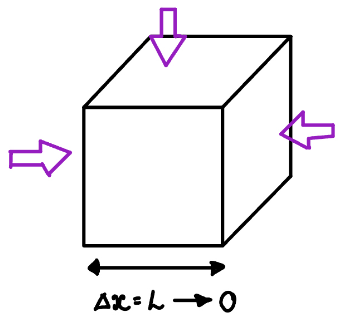



Figure 3: A shrinking box.

**Heuristic argument**: Localise a particle in a box with volume $L^3$, which gets ever smaller such as in fig. 3. Heisenberg's uncertainty principle tells us that

$$\Delta x \Delta p \geq \hbar \tag{15}$$

so where $\Delta x$ is decreasing, $\Delta p$ increases. We can write down the uncertainty in the energy as

$$\Delta E^2 \sim \Delta p^2 c^2 = \left(\frac{\hbar c}{\Delta x}\right)^2. \tag{16}$$

Whenever $\Delta E \sim \frac{\hbar c}{\Delta x} > 2mc^2$ there is enough energy to produce pairs of particles.

Alternatively, take the real life **Large Hadron Collider (LHC)** case. We collide high-energy protons, and produce a shower of lower-energy particles as in fig. 4.

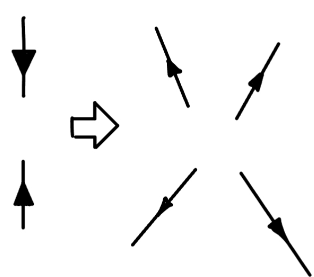


Figure 4: A very simple schematic of a collision at the Large Hadron Collider (LHC).

### 1.2 What can we do with quantum field theory?

A) Treat all particles on the same footing, with a common framework. The *spin-statistics relation "emerges"*.

B) It is language to formulate our most fundamental theorem of nature *parts of which are the most precise & sucessful theory.*

C) It supports a consistent, low energy theory for quantum gravity *our next to last theory.*

### 1.3 How will we build a quantum field theory?

The short answer is by analogy with the theories we know:

A) **Dynamics**: Review Lagrangian & Hamiltonian mechanics, and apply it to fields.

B) **Quantise**: Implement canonical commutation relations.

C) **Interpret the theory we obtain**: Introduce Fock space, and identify familiar operators.

# 2 A: Classical Mechanics Review

## 2.1 Lagrangian Mechanics

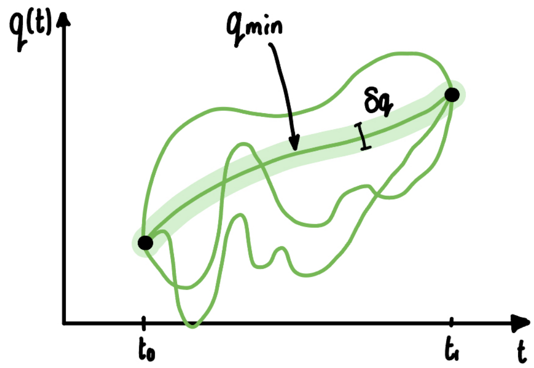



Figure 5: Paths of the coordinate $q$ between fixed $t_0$ and $t_1$. The path which minimises the action is labelled $q_{\mathrm{min}}$. A small deviation around this path is highlighted, and labelled $\delta q$.

We can formulate finding the equations of motion (EoM) of a system as a minimisation problem. Consider some coordinate $q$, with time derivative $\dot{q} = dq/dt$, which is fixed at the endpoints $t_0$ and $t_1$. We want to extremise the action $S$, which is a function of the path between the endpoints, as shown in fig. 5. We can write the action $S$ as the time-integral of the Lagrangian $L$:

$$S = \int dt L(q, \dot{q}) \tag{17}$$

Take $q_{min}(t), \dot{q}_{min}(t)$ as minimising the action. Small "variations" around these functions

$$q = q_{min} + \delta q \tag{18}$$
$$\dot{q} = \dot{q}_{min} + \delta \dot{q} \tag{19}$$

will leave the action invariant (as we're varying around a stationary point)

$$\delta S = \int dt \left( \delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} \right) = 0 \tag{20}$$
$$= \int dt \delta q \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right) + \left( \delta q \frac{\partial L}{\partial \dot{q}} \right|_{t_0}^{t_1} \tag{21}$$
$$\tag{22}$$

where by definition $\delta q = 0$ at the start and end points, so we can ignore the right-most term. We are left with:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 \qquad \text{Euler-Lagrange Equations} \tag{23}$$

## 2.2 Legendre Transform

We will use the Legendre transform to find the inverse of a function $F'(v)$.

### Maths

Take our function $F(v)$ and define $w \equiv \frac{\partial F}{\partial v}$. Then *the inverse of* $w$ *is* $\frac{\partial G}{\partial w}$ *with:*

$$G(w) = w \cdot v(w) - F(v(w)) \tag{24}$$

such that $G(w)$ is the Legendre transform of $F(v)$.

*Proof.*

$$\frac{\partial G}{\partial w}\left(\frac{\partial F}{\partial v}\right) = \cancel{\frac{\partial v}{\partial w}\left(w - \frac{\partial F}{\partial v}\right)} + v$$
$$= v$$

$\square$

### Physics

For us, this is a ticket to Hamiltonian mechanics, i.e. a change of variables from velocity to momentum $(v \to w) \sim (\dot{q} \to p)$ and a change of function from Lagrangian to Hamiltonian $(F \to G) \sim (L \to H)$.

This transformation also has use in statistical mechanics and the path integral formulation of QFT.

### 2.2.1 Hamiltonian Mechanics

First we apply the Legendre transform: treat $\dot{q}$ as independent from $q$ and trade it for $p$ (the canonical momenta) as

$$p \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}} \qquad \left(w = \frac{\partial F}{\partial v}\right) \tag{25}$$

and so we can write the Hamiltonian $H$ in terms of the Lagrangian $L$

$$H(q,p) \equiv p\dot{q}(p,q) - L(q, \dot{q}(p,q)) \tag{26}$$

but the Legendre transformation gives us a first order differential equation

$$\frac{\partial G}{\partial w} = v \qquad \longrightarrow \qquad \frac{\partial H}{\partial p} = \dot{q} \tag{27}$$

Next to obtained a closed set of differential equations, the evolution of $p$ is, :

$$\frac{\partial H}{\partial q} = \frac{\partial \dot{q}}{\partial q}\left(p - \frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = -\dot{p} \tag{28}$$

where we have used eq. (23) (Euler-Lagrange equations) to reach the RHS. We have reached:

$$\boxed{\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \qquad \text{Hamilton's Equations}} \tag{29}$$

We have doubled the degrees of freedom, but halved the order of the differential equations.

### 2.2.2 Example

Consider a point-like mass $m$ in a potential $V$ such as fig. 6.

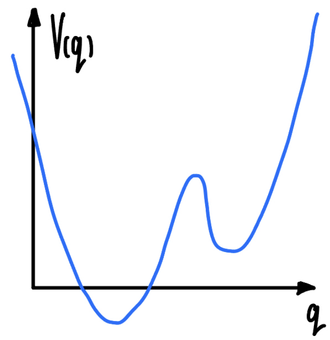



Figure 6: Example potential.

**Lagrangian Formulation**

The Lagrangian is:

$$L = \frac{1}{2}m\dot{q}^2 - V(q). \tag{30}$$

Using the Euler-Lagrange equations, the EoM is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \quad \longrightarrow \quad \frac{d}{dt}m\dot{q} = -V'(q) \tag{31}$$

which is Newton's second law of motion.

**Legendre Transformation**

Let the momentum $p = m\dot{q}$. Then we can write the Hamiltonian using Eq. 26

$$\begin{aligned} H &= p \cdot \dot{q} - \frac{1}{2}m\dot{q}^2 + V(q) && (32)\\ &= \frac{p^2}{m} - \frac{p^2}{2m} + V(q) && (33)\\ &= \frac{p^2}{2m} + V(q). && (34) \end{aligned}$$

And using Hamilton's equations Eq. 29 for the EoM

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad\qquad \dot{p} = -\frac{\partial H}{\partial q} = -V' \tag{35}$$

We see that nothing is lost, or gained. These are equivalent formulations.

> Incidentally
> $$\frac{\partial L}{\partial \dot{q}}\Big|_{\dot{q}=y} = my \quad \xleftrightarrow{\text{inverse}} \quad \frac{\partial H}{\partial p}\Big|_{p=y} = \frac{y}{m} \tag{36}$$

## 2.3 Poisson Brackets and Time Evolution

Let's look for constants of motion. Consider an arbitrary function $\mathcal{O} = \mathcal{O}(q, p, t)$.

$$\frac{d\mathcal{O}(q,p,t)}{dt} = \dot{q}\frac{\partial \mathcal{O}}{\partial q} + \dot{p}\frac{\partial \mathcal{O}}{\partial p} + \frac{\partial \mathcal{O}}{\partial t} \tag{37}$$

$$= \frac{\partial H}{\partial p}\frac{\partial \mathcal{O}}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial \mathcal{O}}{\partial p} + \frac{\partial \mathcal{O}}{\partial t} \tag{38}$$

where we have used the equations of motion to go from the first to second line.

We define a **Poisson bracket** as

$$[A, B]_P = \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} \tag{39}$$

such that

$$\frac{d\mathcal{O}}{dt} = [H, \mathcal{O}] + \frac{\partial \mathcal{O}}{\partial t} \tag{40}$$

then we have that

- Any magnitude $\mathcal{O}$ such that
$$[H, \mathcal{O}] = -\frac{\partial O}{\partial t} \tag{41}$$
is conserved.
- For $H$ time-independent, $H$ itself is conserved.
- This resembles QM and Heisenberg's picture.

## 2.4 Classical Field Mechanics [PS 2.2, S 3, T 1.1]

Formally, the number of degrees of freedom will blow up now, since there is one per space-point $\phi(t, \vec{x})$, so space is somewhat like time but not quite. If this sounds funny, you've seen this before:

$$\vec{E}(\vec{x}), \vec{B}(\vec{x}) \quad ; \quad \Phi(\vec{x}), \vec{A}(\vec{x}) \tag{42}$$

$$\vec{E} = -\nabla\Phi - \frac{\partial}{\partial t}\vec{A} \quad ; \quad \vec{B} = \nabla \times \vec{A} \tag{43}$$

The change is then

$$\begin{array}{ccc} \mathbb{R} \to \mathbb{R} & q(t) \longleftrightarrow \phi(t,x) & \mathbb{R}^{n+1} \to \mathbb{R} \\ \mathbb{R} \to \mathbb{R} & p(t) \longleftrightarrow \Pi(t,x) & \mathbb{R}^{n+1} \to \mathbb{R} \\ & t \longleftrightarrow t; x & \end{array}$$

We can picture the field as a "mattress", like in fig. 7 (see Zee).

### 2.4.1 Lagrangian Formulation

Sure, the Lagrangian formulation should do, but what are we to make of the extra label? Let $\dot{\phi} = \frac{\partial \phi}{\partial t}$, then

$$S = \int dt L(\phi, \dot{\phi}, \nabla\phi, ...) \tag{44}$$

$$= \int dt \left( \int d^3x A(\phi(x)) + \int d^3x d^3y B\left(\phi(x), \phi(y)\right) + ... \right) \tag{45}$$

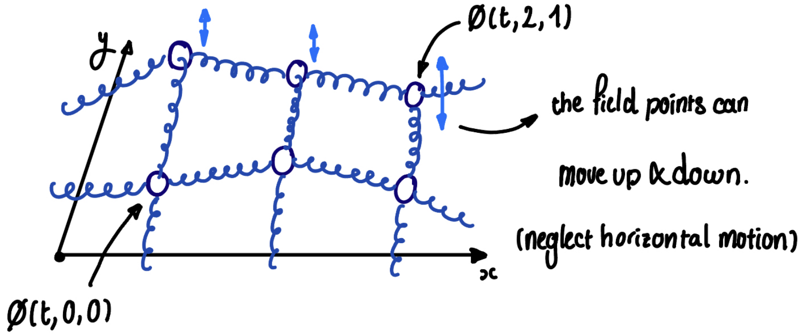



Figure 7: Zee's "mattress" visualisation of a field.

But remember locality! The field at $\phi(t,0)$ does not have it's dynamics influenced directly and instantaneously by the field at $\phi(t,z)$. As such, $B(\phi(x),\phi(y)) = 0$ and instead we introduce the *Lagrangian Density*, $\mathcal{L}$.

$$S = \int dt \int \mathcal{L}(\phi, \dot{\phi}, \nabla\phi) \tag{46}$$

including boundary conditions (BC). A small deviation of the action, shown in fig. 8, is given by

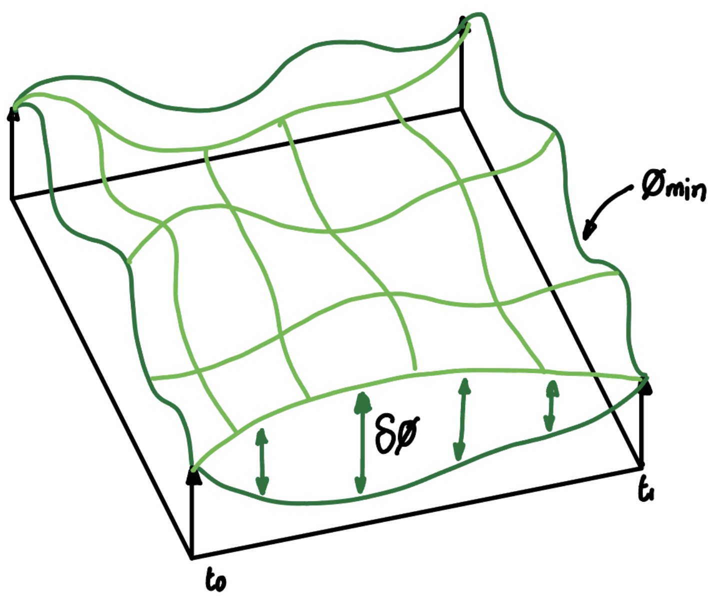



Figure 8: A small variation of the field $\phi$, $\delta\phi$ in dark green from that which minimises the action $\phi_{\min}$ shown in lighter green.

$$\delta S = \int dt d^3x \left( \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta\dot{\phi} \frac{\mathcal{L}}{\partial \dot{\phi}} + \frac{\partial \delta\phi}{\partial x^i} \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \phi}{\partial x^i} \right)} \right) \tag{47}$$

$$= \int dt d^3x \delta\phi \left( \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x^i}} \right) \tag{48}$$

$$+ \int \delta\phi \frac{\partial \mathcal{L}}{\partial \dot{\phi}} d^3x \Big|_{t_0}^{t_1} + \int dt \delta\phi \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x^i}} \cdot d\vec{\Sigma} \tag{49}$$

where, $d\vec{\Sigma}$ is an infinitesimal element normal to the surface we integrate over and on the final line, the first term cancels for Dirichlet boundary conditions, and we assume that $\phi \to 0$ which sends the second term to zero. Note if we're in finite space, there is a choice of Neumann or Dirichlet boundary conditions. We can now extract field equations

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x^i}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \qquad \text{Field Equations} \tag{50}$$

where time and space are now on the same footing. We can package them up into a 4-vector $x^\mu = (ct, \vec{x})$, and rewrite the field equations:

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \phi}{\partial x^\mu} \right)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{51}$$

This formula holds regardless of Lorentz invariance.

### 2.4.2 Hamiltonian Formulation

The Hamiltonian formalism for fields now has variations where we had derivatives:

$$\int d^3x \delta\dot{\phi}\, \Pi \equiv \int d^3x \delta\dot{\phi} \frac{\partial \mathcal{L}}{\delta \dot{\phi}} \tag{52}$$

$$= \int d^3x \delta\dot{\phi} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \nabla \frac{\partial \mathcal{L}}{\partial \nabla \dot{\phi}} \right) \tag{53}$$

such that the canonical coordinate is

$$\Pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \nabla \frac{\partial \mathcal{L}}{\partial \nabla \dot{\phi}} \tag{54}$$

where in almost all cases we will consider, the right-most term will vanish. Now we can write the Hamiltonian as

$$H = \int d^3x \mathcal{H} = \int d^3x \left( \Pi \cdot \dot{\phi} - \mathcal{L} \right) \qquad \text{Hamiltonian} \tag{55}$$

The equations of motion now follow from Legendre's inverse transform such that

$$\int d^3x \delta\phi \frac{\delta \mathcal{H}}{\delta \phi} = \int d^3x \left( \frac{\delta \dot{\phi}}{\delta \phi} \left( \Pi - \frac{\delta \mathcal{L}}{\delta \dot{\phi}} \right) - \frac{\delta \mathcal{L}}{\delta \phi} \right) \tag{56}$$

as

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial \Pi} - \nabla \frac{\partial \mathcal{H}}{\partial \nabla \Pi} \qquad\qquad \dot{\Pi} = -\frac{\partial \mathcal{H}}{\partial \phi} + \nabla \frac{\partial \mathcal{H}}{\partial \nabla \phi} \tag{57}$$

Poisson's bracket has a generalisation such that

$$\frac{d}{dt}\Theta = [H, \Theta]_P + \frac{\partial \Theta}{\partial t} \tag{58}$$

#### 2.4.3 Example: Klein-Gordon Field

We are headed, eventually, to our theory of elementary particles; in this context the Klein-Gordon field is fundamental & including interactions will describe the Higgs boson. Here it is, it's Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\left[\dot{\phi}^2 - (c_\phi \vec{\nabla}\phi)^2 - \alpha\phi^2\right] \tag{59}$$

where $c_\phi, \alpha$ are constants. The EoM is given by:

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \vec{\nabla}\frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{60}$$

$$\Rightarrow \ddot{\phi} - c_\phi^2 \vec{\nabla}\phi + \alpha\phi \equiv (\Box + \alpha)\phi = 0 \tag{61}$$

The Hamiltonian density follows from the canonical coordinate

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \tag{62}$$

such that

$$\mathcal{H} = \Pi\dot{\phi} - \frac{1}{2}\left[\Pi^2 - (c_\phi \vec{\nabla}\phi)^2 - \alpha\phi^2\right] \tag{63}$$

$$= \frac{1}{2}\left(\Pi^2 + (c_\phi \vec{\nabla}\phi)^2 + \alpha\phi^2\right) \tag{64}$$

#### 2.4.4 Example: Low Energy Accoustic Phonons

A small modification of the Klein-Gordon Lagrangian allows us to describe low-energy phonons.

Take $\mathcal{D}(x)$ to be the displacement of an atom of mass $m$ interacting with it's neighbours via a spring (see fig. 9.

$$\mathcal{L} = \frac{1}{2}\left(m\dot{\mathcal{D}}^2 - \mathcal{K}(l\nabla\mathcal{D})^2 + \frac{\mathcal{K}}{4}(l^2\vec{\nabla}\mathcal{D})^2\right) \tag{65}$$

with a Hamiltonian density

$$\mathcal{H} = \frac{1}{2m}\Pi^2 + \frac{\mathcal{K}}{2}\left[(l\vec{\nabla}\mathcal{D})^2 - \frac{(l^2(\vec{\nabla})^2\mathcal{D})^2}{4}\right] \tag{66}$$

#### 2.4.5 Example: Electromagnetism

It's as simple as

$$\mathcal{L}_{EM} = \frac{\varepsilon_0}{2}\left(\vec{E}^2 - c^2\vec{B}^2\right) \tag{67}$$

$$= \frac{\varepsilon_0}{2}\left(\left(\delta\nabla\Phi + \partial_t\vec{A}\right)^2 - c^2\left(\vec{\nabla}\times\vec{A}\right)^2\right) \tag{68}$$

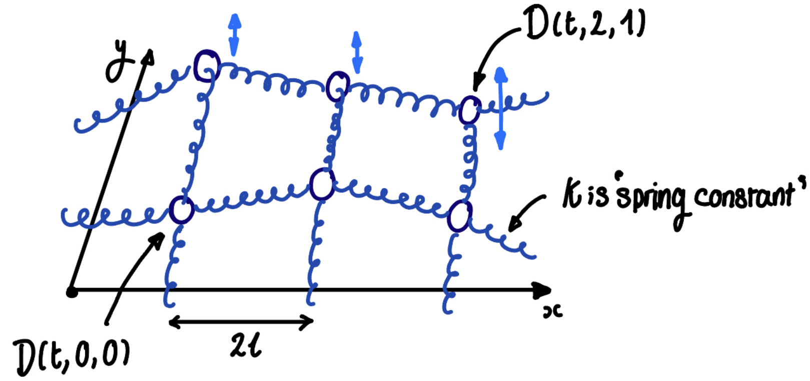



Figure 9: A schematic of atoms of mass $m$ interacting with it's neighbours with springs of spring constant $\mathcal{K}$. The displacement of the atom at the $x^{\text{th}}$ and $x^{\text{th}}$ position at time $t$ is labelled as $\mathcal{D}(t,x,y)$.

where we note that $\vec{E}$ is the canonical momenta of $\vec{A}$, and we have used eqs. (11) and (12). We will find it useful to rewrite the Lagrangian in terms of the field strength tensor

$$F_{\mu,\nu} \equiv \begin{pmatrix} 0 & \left(\vec{\nabla}\Phi + \partial_t \vec{A}\right)^T \\ -\vec{\nabla}\Phi - \partial_t \vec{A} & c\left(-\nabla_i A^j + \nabla_j A^i\right) \end{pmatrix} \tag{69}$$

where we have used that $\mu \in [0, i]$ and $i = 1, 2, 3$. Recall the Minkowski metric, defined in section 0.2.1 can be used to raise and lower indices. Then we can rewrite the Lagrangian density as

$$\mathcal{L}_{EM} = +\frac{\varepsilon_0}{4} Tr(\eta F \eta F) \tag{70}$$

### 2.4.6 Example: Complex, Constrained Field

One last example is a complex field $\varphi$ (conjugate is $\varphi^*$), with a first order EoM

$$\mathcal{L} = i\hbar\varphi^*\partial_t\varphi - \frac{\hbar^2\vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi}{2m} \tag{71}$$

we treat $\varphi, \varphi^*$ as independent so that the EoM are

$$\partial_t \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^*} + \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\varphi^*} - \frac{\partial \mathcal{L}}{\partial \varphi^*} = -\frac{\hbar^2\vec{\nabla}^2\varphi}{2m} - i\hbar\dot{\varphi} = 0 \tag{72}$$

$$\partial_t \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} + \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\varphi} - \frac{\partial \mathcal{L}}{\partial \varphi} = i\hbar\dot{\varphi}^* - \frac{\hbar^2\vec{\nabla}^2\varphi^*}{2m} = 0 \tag{73}$$

the canonical momenta is

$$\Pi = i\varphi^*\hbar\Pi^* = 0 \tag{74}$$

and so we find the Hamiltonian density is

$$\mathcal{H} = \Pi\dot{\varphi} + 0 \cdot \dot{\varphi}^* - \left(\Pi\dot{\varphi} - \frac{\hbar\vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi}{2m}\right) \tag{75}$$

$$= \frac{\hbar\vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi}{2m} \tag{76}$$

$$= \frac{i\hbar\vec{\nabla}\Pi \cdot \vec{\nabla}\varphi}{2m} \tag{77}$$

# 3 B: Quantisation [P.S. 2.3, S 2.3, Z 1.8, T 2.1-2.4]

Thus far, we have reviewed classical mechanics and even seen how, with Poisson brackets, the evolution of observables is akin to that of quantum mechanics. To quantise the theory, however, we need to input the uncertainty principle. Again, parallels help:

$$[X, P] = i\hbar \Rightarrow [\phi, \Pi] = i\hbar\text{“}1\text{”} \tag{78}$$

This tells us we cannot measure momentum and position simultaneously, and to arbitrary precision. When we translate to canonical variables, we cannot determine the field value and it's time derivative to arbitrary precision either. This is our starting point: we promote $\phi, \Pi$ to operators and impose a form that satisfies commutation relations.

## 3.1 The Quantum Harmonic Oscillator

### 3.1.1 Review

The way we will go about quantisation calls for a review of the harmonic oscillator first. Take a Hamiltonian

$$H = \frac{1}{2}\frac{P^2}{2m} + \frac{m\omega^2 X^2}{2} \tag{79}$$

where $X$ and $P$ are position and momentum operators respectively. We introduce creation $a^\dagger$ and annihilation $a$ operators, such that $[a, a^\dagger] = 1$ and we write an ansatz for $X$ and $P$ as

$$X = (a + a^\dagger)C \tag{80}$$

$$P = i(a - a^\dagger)B\hbar \tag{81}$$

which satisfies the commutation relations

$$[X, P] = i\hbar([a, -a^\dagger] + [a^\dagger, a])CB = -2i\hbar CB \tag{82}$$

where we can take $C, B$ such that

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \tag{83}$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger) \tag{84}$$

which give us $[X, P] = i\hbar$, but also

$$H = \left(\frac{m}{2m}\left(-\frac{\omega}{2}\right)\left(a^2 + (a^\dagger)^2 - \{a, a^\dagger\}\right)\right)\hbar \tag{85}$$

$$+ \frac{\omega^2 m}{4m\omega}\left(a^2 + (a^\dagger)^2 + \{a, a^\dagger\}\right)\hbar \tag{86}$$

$$= \frac{\hbar\omega}{2}(\{a, a^\dagger\}) \tag{87}$$

$$= \omega\left(aa^\dagger + \frac{1}{2}\right)\hbar \tag{88}$$

where $\{a, a^\dagger\} = aa^\dagger + a^\dagger a$, i.e. the anti-commutator, so that

$$[H, a^\dagger] = \hbar\omega a^\dagger[a, a^\dagger] = \hbar\omega a^\dagger \tag{89}$$

$$[H, a] = \hbar\omega[a^\dagger, a]a = -\hbar\omega a \tag{90}$$

The action of $a^\dagger$ ($a$) on a Hamiltonian eigenstate is to increase (decrease) the energy by $\omega$. Given the positive definite spectrum, there must be a state $|0\rangle$ such that $a|0\rangle = 0$ and the whole spectrum is spanned by the Hilbert space:

$$\bigoplus (a^\dagger)^n |0\rangle \tag{91}$$

which has the following promising features:

- Energy is $\sim \sqrt{\omega^2}$ and we have a positive spectrum.
- Creation and annihilation operators offer a simple picture of quanta.

### 3.1.2 Quantum Harmonic Oscillator Field

For fields now we should specify what happens to the labels $\vec{x}, \vec{y}$ on the right hand side of the commutator. In our mattress picture fig. 7, there's an oscillator at every $\vec{x}$ so we have that the commutator is only $i\hbar$ for $\phi, \Pi$ on the same side:

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\hbar\delta^3(\vec{x} - \vec{y})\mathbb{I} \tag{92}$$

where $\mathbb{I}$ refers to the identity matrix. We now try the same type of ansatz as for $X$ and $P$ above:

$$\phi(\vec{x}) = \int [d\vec{k}] A \left( a_k e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}} \right) \tag{93}$$

$$\Pi(\vec{x}) = -i \int [d\vec{k}'] B \left( a_{k'} e^{i\vec{k}'\cdot\vec{x}} - a_{k'}^\dagger e^{-i\vec{k}'\cdot\vec{x}} \right) \tag{94}$$

with the notation

$$[d\vec{k}] = \frac{d^3k}{(2\pi)^3 N_k} \qquad , \qquad [a_k, a_{k'}^\dagger] = (2\pi)^3 N_k \delta^3(\vec{k} - \vec{k}') \tag{95}$$

with $\vec{k}, \vec{k}'$ wavevectors of units $1/[length]$.

$N_k$ is a normalisation that differs in different texts. It does not matter for physical results. We note that

$$\int [d\vec{k}] f(k, k')[a_{k'}, a_k^\dagger] = f(k, k) \tag{96}$$

Now let's look at the commutator

$$[\phi(\vec{x}), \Pi(\vec{y})] \tag{97}$$

$$= -i \int [d\vec{k}][d\vec{k}'] AB \Big( e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{y}} [a_k, -a_{k'}^\dagger] \tag{98}$$

$$+ e^{-i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{y}} [a_k^\dagger, a_k'] \Big) \tag{99}$$

$$= +i \int [d\vec{k}] AB \left( e^{i\vec{k}(\vec{x}-\vec{y})} + h.c. \right) = i\hbar\delta^3(\vec{x} - \vec{y}) \tag{100}$$

such that $\frac{2A\cdot B}{N_k} = \hbar$ after expanding our notation. Now the Hamiltonian will dictate the energy relation to momentum and give us the spectrum of the theory, which should be bounded from below on energy.

### 3.1.3 Example: Klein-Gordon Field

To be explicit, let's use the Klein-Gordon field, which will also show us the way to a consistent, relativistic theory. The EoM is

$$\left(\frac{\partial^2}{\partial t^2} - c_\phi^2\nabla^2 + \alpha\right)\phi(x) = 0 \tag{101}$$

which resembles the equation we got by "squaring" the dispersion relation in Eq. 60, but recall when applied on the wavefunction gave $E \to -\infty$ states. Here however, $\phi$ **is a field** and we shall hold our conclusions until we have studied the Hamiltonian

$$H = \int d^3x \frac{1}{2}\left(\Pi^2 + (c_\phi\vec{\nabla}\phi)^2 + \alpha\phi^2\right) \tag{102}$$

where we substitute our ansatz from earlier, eqs. (93) and (94)

$$\mathcal{H} = -\frac{B^2}{2}\int [d\vec{k}][d\vec{k}'] \left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_{k'} e^{i\vec{k}'\cdot\vec{x}} - a_{k'}^\dagger e^{-i\vec{k}'\cdot\vec{x}}\right) \tag{103}$$

$$+ \frac{A^2c_\phi^2}{2}\int [d\vec{k}][d\vec{k}'] \left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right) i\vec{k}\cdot i\vec{k}' \left(a_{k'} e^{i\vec{k}'\cdot\vec{x}} - a_{k'}^\dagger e^{-i\vec{k}'\cdot\vec{x}}\right) \tag{104}$$

$$+ \frac{\alpha A^2}{2}\int [d\vec{k}][d\vec{k}'] \left(a_k e^{i\vec{k}\cdot\vec{x}} + h.c.\right)\left(a_{k'} e^{i\vec{k}'\cdot\vec{x}} + h.c.\right) \tag{105}$$

If we collect terms and integrate over $d^3x$, we can use that $\int d^3x e^{i\vec{x}(\vec{k}-\vec{k}')} = (2\pi)^3\delta^3(\vec{k}-\vec{k}')$ and we obtain

$$H = \int [d\vec{k}]\left(-B^2 + A^2c_\phi^2k^2 + A^2\alpha\right)\frac{a_k a_{-k}}{2}\frac{1}{N_k} + h.c. \tag{106}$$

$$+ \int [d\vec{k}]\left(B^2 + A^2c_\phi^2k^2 + A^2\alpha\right)\frac{\{a_k, a_k^\dagger\}}{2}\frac{1}{N_k} \tag{107}$$

$$\stackrel{!}{=} \int [d\vec{k}] E_k \frac{\{a_k, a_k^\dagger\}}{2} \tag{108}$$

where by forcing the form of $H$ in the final line, we find a set of equations ($\vec{k} = \vec{p}/\hbar$)

(1) $$B^2 = A^2(k^2c_\phi^2 + \alpha)$$

(2) $$\frac{2AB}{N_k} = \hbar$$

(3) $$\frac{B^2 + A^2(k^2c_\phi^2 + \alpha)}{N_k} = E_k$$

Noting that since $N_k > 0$, energy $E_k > 0$ also. The normalisation $N_k$ is a given value (commonly $N_k = 1$ such as in Peskin & Schroeder, and Tong; also $N_k = 2E_k$), so we can solve for the 3 unknowns: $E_k, A, B$. The solutions are

$$E_k = \hbar\sqrt{k^2c_\phi^2 + \alpha} \stackrel{\text{more relativistic E}}{=} \sqrt{p^2c^2 + m^2c^4} \tag{109}$$

$$A^2 = \frac{\hbar N_k}{2}\sqrt{k^2c^2 + \alpha} = \frac{\hbar^2 N_k}{2E_k} \tag{110}$$

$$B^2 = \frac{\hbar N_k}{2}\sqrt{k^2c^2 + \alpha} = \frac{N_k E_k}{2} \tag{111}$$

We have found the field that presents the **relativistic energy relation** (provided that $c_\phi = c$, the speed of light, and $\alpha = m^2c^4/\hbar^2$) and has a **positive definite spectrum** as

$$H = \int [d\vec{k}] E_k \left( a_k^\dagger a_k + \frac{(2\pi^3) N_k \delta^3(0)}{2} \right) \tag{112}$$

$$E_k = \hbar\sqrt{\vec{k}^2c^2 + \alpha} = \sqrt{\vec{p}^2c^2 + m^2c^4} \tag{113}$$

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 N_k \delta^3(\vec{k} - \vec{k}') \tag{114}$$

$$a_k |0\rangle \equiv 0 \qquad \text{where } |0\rangle \text{ is the vacuum} \tag{115}$$

This field quantum harmonic oscillator system is the basis for QFT and our ansatz for the fields is

$$\phi(\vec{x}) = \hbar \int [d\vec{k}] \sqrt{\frac{N_k}{2E_k}} \left( a_k e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}} \right) \tag{116}$$

$$= \hbar \int \frac{d^3k}{(2\pi)^3 \sqrt{N_k 2E_k}} \left( a_k e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}} \right) \tag{117}$$

$$\Pi(\vec{x}) = -i \int [d\vec{k}] \sqrt{N_k} 2E_k E_k \left( a_k e^{i\vec{k}\cdot\vec{x}} - a_k^\dagger e^{-i\vec{k}\cdot\vec{x}} \right) \tag{118}$$

$$= -i \int \frac{d^3k E_k}{(2\pi)^3 \sqrt{N_k 2E_k}} \left( a_k e^{i\vec{k}\cdot\vec{x}} - a_k^\dagger e^{-i\vec{k}\cdot\vec{x}} \right) \tag{119}$$

### 3.1.4 The Quanta Of The Field

What is specific to a Klein-Gordon field is the energy relation $E_k$, but we can apply the rest to our other examples. A common feature is that the quanta created by $a^\dagger$ has a particle interpretation

$$[H, a_{k'}^\dagger] = \int [d\vec{k}] [a_k^\dagger a_k, a_{k'}^\dagger] E_k = E_{k'} a_{k'}^\dagger$$

so that if we act on an energy eigenstate $|E_s\rangle$ with energy $E_s$, with the Hamiltonian $H$:

$$|E_{s'}\rangle = a_k^\dagger |E_s\rangle \qquad H|E_{s'}\rangle = \left( [H, a_{k'}^\dagger] + a_{k'}^\dagger H \right) |E_s\rangle \tag{120}$$

$$= (E_k + E_s) |E_{s'}\rangle . \tag{121}$$

I.e. what is returned is another energy eigenstate, with energy increased by a quanta $E_k$ from when $a^\dagger$ acts. On the other hand, $a$ decreases $E_s$ and so we are led to the conclusion that

$$a_k^\dagger \leftrightarrow \text{creates particle with momentum } \vec{p} = \hbar\vec{k} \tag{122}$$

$$a_k \leftrightarrow \text{annihilates particle with momentum } \vec{p} = \hbar\vec{k} \tag{123}$$

Note that the $\delta^3(0)$ term does not affect $[H, a^\dagger]$, i.e. the difference of energy between states, and is present even for $|0\rangle$; it is a vacuum energy & an overall shift of all energies. Here we are interested only in energy differences and so we will drop it. Side note, that it relates to one of the deepest puzzles in physics, the cosmological constant $\Lambda$.

To drop this term, we introduce **normal ordering** $: O :$ where

$$: a_{k_1} a_{k_2}^\dagger : = a_{k_2}^\dagger a_{k_1}$$

$$: a_{k_1} a_{k_2} a_{k_3}^\dagger a_{k_4} a_{k_5}^\dagger : = a_{k_3}^\dagger a_{k_5}^\dagger a_{k_1} a_{k_2} a_{k_4}$$

That is, all $a^\dagger$'s left of the $a$'s, so that:

$$H \equiv: \frac{1}{2}\int d^3x \left(\Pi^2 + (c\nabla\phi)^2 + \alpha\phi^2\right) : \tag{124}$$

$$= \int [d\vec{k}] E_k : \frac{\{a_k^\dagger, a_k\}}{2} : \tag{125}$$

$$= \int [d\vec{k}] E_k a_k^\dagger, a_k \tag{126}$$

With our normal ordered definition of $H$, $H|0\rangle = 0$. Next, is the first excited state with $Ha_k^\dagger|0\rangle = E_k a_k^\dagger|0\rangle$ where $E_k \geq mc^2$. Next, a two-particle state $a_k^\dagger a_{k'}^\dagger|0\rangle$ and so on. A general state in this space will be a superposition of such states as

$$|s\rangle = f^{(0)}|0\rangle + f^{(1)}(k_1)a_{k_1}^\dagger|0\rangle + f^{(2)}(k_1,k_2)a_{k_1}^\dagger a_{k_2}^\dagger|0\rangle \tag{127}$$

$$= \sum_n f^{(n)}(k_1,...,k_n)\left(\prod_n a_{k_i}^\dagger\right)|0\rangle \tag{128}$$

with

$$H(a_{k_1}^\dagger...a_{k_n}^\dagger)|0\rangle = \left(\sum_i E_{k_i}\right)a_{k_1}^\dagger...a_{k_n}^\dagger|0\rangle \tag{129}$$

This space is larger than your usual Hilbert space & its dimensions are hard to grasp; it is, however, the space we were looking for, a space that includes multiparticle states and will allow us to describe transitions. **It is called Fock Space**.

**Normalisation of momentum states**

- *In a box*: If in a box of volume $V$ we have discretised momentum and

$$(2\pi)^3\delta^3(\vec{k}-\vec{k}') \to \int_V d^3x e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \xrightarrow{\vec{k}\to\vec{k}'} V \tag{130}$$

  so that

$$|k\rangle_V \equiv \frac{a_k}{\sqrt{N_k V}}|0\rangle \qquad\qquad {}_V\langle k|k\rangle_V = 1 \tag{131}$$

- *Non-Relativistic*: We use arrow notation $\left|\vec{k}\right\rangle$

$$\left\langle\vec{k}\middle|\vec{k}\right\rangle = (2\pi)^3\delta^3(\vec{k}-\vec{k}') \tag{132}$$

$$\left|\vec{k}\right\rangle \equiv \frac{a_k}{\sqrt{N_k}}|0\rangle \tag{133}$$

- *Relativistic*: Now without an arrow $|k\rangle$

$$\langle k|k\rangle = (2\pi)^3\delta^3(\vec{k}-\vec{k}') \tag{134}$$

$$|k\rangle \equiv \sqrt{\frac{2E_k}{N_k}}a_k|0\rangle \tag{135}$$

  also note

$$\phi(\vec{x})|0\rangle = \int [d\vec{k}]e^{-i\vec{k}\cdot\vec{x}}\sqrt{\frac{2E_k}{N_k}}a_k^\dagger|0\rangle \tag{136}$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k}e^{-i\vec{k}\cdot\vec{x}}|k\rangle \tag{137}$$

### 3.1.5 Complex Klein Gordon Field

The motions are the same if we have a complex scalar field with

$$\mathcal{L} = \dot{\phi}^\dagger\dot{\phi} - c^2\vec{\nabla}\phi^\dagger\vec{\nabla}\phi - \alpha\phi^\dagger\phi \tag{138}$$
$$\mathcal{H} = \dot{\phi}^\dagger\dot{\phi} + c^2\vec{\nabla}\phi^\dagger\vec{\nabla}\phi + \alpha\phi^\dagger\phi \tag{139}$$

so now we propose a non-hermitian ansatz

$$[a_k, a^\dagger_{k'}] = [b_k, b^\dagger_{k'}] = (2\pi)^3 N_k \delta^3(\vec{k} - \vec{k}') \tag{140}$$
$$\phi = \int [d\vec{k}] A \left(a_k e^{i\vec{k}\cdot\vec{x}} + b^\dagger_k e^{-i\vec{k}\cdot\vec{x}}\right) \tag{141}$$
$$\Pi = i\int [d\vec{k}] B \left(a^\dagger_k e^{-i\vec{k}\cdot\vec{x}} - b_k e^{i\vec{k}\cdot\vec{x}}\right) \tag{142}$$

and, as you're asked to show yourselves,

$$:\int d^3x\mathcal{H}: = \int [d\vec{k}](a^\dagger a + b^\dagger b)E_k \tag{143}$$
$$\phi = \int [d\vec{k}]\sqrt{\frac{N_k}{2E_k}} \left(a_k e^{i\vec{k}\cdot\vec{x}} + b^\dagger_k e^{-i\vec{k}\cdot\vec{x}}\right) \tag{144}$$
$$\Pi = i\int [d\vec{k}]\sqrt{\frac{N_k E_k}{2}} \left(a^\dagger_k e^{-i\vec{k}\cdot\vec{x}} - b_k e^{i\vec{k}\cdot\vec{x}}\right) \tag{145}$$

### 3.1.6 Example: Complex, Constrained Field

At times, our system will have constraints which reduce the degrees of freedom. Let's see a case:

$$\mathcal{L} = \hbar\varphi^\dagger i\partial_t\varphi - \frac{\hbar^2\nabla\varphi^\dagger\nabla\varphi}{2m} \tag{146}$$
$$\mathcal{H} = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}}\dot{\varphi} + \frac{\delta\mathcal{L}}{\delta\dot{\varphi}^\dagger}\dot{\varphi}^\dagger - \mathcal{L} = \frac{\hbar^2\nabla\varphi^\dagger\nabla\varphi}{2m} \tag{147}$$

The key is in phase-space dimension, the momentum is not independent as $\Pi = i\varphi^\dagger$. Let's recklessly push ahead nonetheless:

$$\varphi = \int [d\vec{k}] \left(A a_k e^{i\vec{k}\cdot\vec{x}} + B b^\dagger_k e^{-i\vec{k}\cdot\vec{x}}\right) \tag{148}$$
$$\Pi = i\int [d\vec{k}'] \left(A^* a^\dagger_{k'} e^{-i\vec{k}'\cdot\vec{x}} + B^* b_k e^{i\vec{k}'\cdot\vec{x}}\right) \tag{149}$$

Hamiltonian

$$H \propto ik' \left(A^* a^\dagger_{k'} e^{i\vec{k}'\cdot\vec{x}} - B^* b_{k'} e^{-i\vec{k}'\cdot\vec{x}}\right) \times ik \left(A a_{k'} e^{i\vec{k}'\cdot\vec{x}} - B b^\dagger_k e^{-i\vec{k}'\cdot\vec{x}}\right) \tag{150}$$

where $AB = A^*B^* = 0$. We are forced to give up one degree of freedom! This is what phase space was trying to tell us, instead of the two d.o.f. of a complex field, this system has one only. We will see more versions of this & its interplay with Lorentz invariance. For now we conclude:

$$H = \int [d\vec{k}]\frac{\hbar^2 k^2}{2m} a^\dagger_k a_k \tag{151}$$
$$\varphi = \int [d\vec{k}]\frac{N_k}{e}^{i\vec{k}\cdot\vec{x}} a_k \tag{152}$$

In passing, we note that this, which gives a non-relativistic particle Fock space, is a QFT version of Schroedinger's equation. The fact that field & wavefunction have the same equation is partially to blame for the name "second" quantisation.

## 3.2 Fock Space and Operators [PS 2.4, S 2.3, Z 1.8, T 2.4-2.6]

Whatever our classical field theory, after quantisation, we have our Hamiltonian & Fock space. This is all we need to lay out the expectation for a given observable represented by a Hermitian operator $\mathcal{O}$.

**Expectation value of $\mathcal{O}$ at a time t**:

$$\begin{array}{cc} \text{Schroedinger} & \text{Heisenburg} \\ \langle s,t|\mathcal{O}(\vec{x})|s,t\rangle & \langle s|\mathcal{O}(t,\vec{x})|s\rangle \\ i\hbar\frac{\partial}{\partial t}\left|s,t\right\rangle = H\left|s,t\right\rangle & i\hbar\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O},H] \end{array}$$

where now position is a label and the two pictures are equivalent. They will come in handy at different stages.

It is useful to define the evolution operator

$$\left|s,t\right\rangle = U_0(t,0)\left|s,0\right\rangle \tag{153}$$

$$\mathcal{O}(t,\vec{x}) = U_0^\dagger(t,0)\mathcal{O}(0,\vec{x})U_0(t,0) \tag{154}$$

so both pictures give

$$\langle s,0|U_0^\dagger(t,0)\mathcal{O}(0,\vec{x})U_0(t,0)|s,0\rangle \tag{155}$$

For our quadratic harmonic-oscillator-like theories, evolution is simple

$$i\hbar\partial_t U_0 = HU_0 \qquad\qquad U_0(t) = -e^{-iHt/\hbar} \tag{156}$$

let us then evolve our first operator

$$a_k(t) = e^{iHt/\hbar}a_k e^{-iHt/\hbar} \tag{157}$$

$$= a_k + \frac{it}{\hbar}\int [dk']E_{k'}\left[a^\dagger_{k'}a_{k'},a_k\right] + \mathcal{O}(H^2) \tag{158}$$

$$= a_k + \frac{it}{\hbar}\int [dk']E_{k'}\left[a^\dagger_{k'},a_k\right]a_{k'} + \mathcal{O}(H^2) \tag{159}$$

$$= a_k - \frac{iE_k t}{\hbar}a_k + \mathcal{O}(H^2) \tag{160}$$

For higher terms, use the Baker-Campbell-Hausdorff formula

$$e^B A e^{-B} = \sum_n \frac{1}{n!}\overbrace{[B[B[...[B,}^{\text{n times}}A]]] \tag{161}$$

You can show that

$$[H,[H,a_k]] = [H,-iE_k a_k] = -E_k^2 a_k \tag{162}$$

$$a_k(t) = e^{-iE_k t/\hbar}a_k \tag{163}$$

which carries over to

$$\Phi_I(t,\vec{x}) = \hbar\int[d\vec{k}]\sqrt{\frac{N_k}{2E_k}}\left(e^{-i\omega_k t + i\vec{k}\cdot\vec{x}}a_k + h.c.\right) \tag{164}$$

the label $I$, just like $U_0$, seems odd to add now but will be useful when talking about interactions.

> One can also obtain this solution from the Euler-Lagrange equations for the operator
>
> $$(\partial_t^2 - c^2\nabla^2 + \alpha)\Phi_i(t, \vec{x}) = 0 \tag{165}$$

Let's compute observables. We sort of already did, since we know:

$$\langle 0|a_k H a_k^\dagger|0\rangle = E_k \langle 0|a_k a_k^\dagger|0\rangle \tag{166}$$

$$= E_k (2\pi)^3 N_k \delta^3(\vec{k} - \vec{k}) \tag{167}$$

The infinity has to do with normalisation, but setting that aside, the Hamiltonian expectation value is the energy, it's squared will be $E_k^2$ etc, as it should be for an eigenstate. What about the field itself?

$$\langle 0|a_k \Phi_I(t, \vec{x}) a_k^\dagger|0\rangle = \int [d\vec{k}] A \Big( \langle 0|a_k \left( a_{k'} e^{-ik'\cdot x} + a_{k'}^\dagger e^{ik'\cdot x} a_k^\dagger \right) |0\rangle \Big) \tag{168}$$

$$= \int [d\vec{k}] A \Big( e^{-ik'x} \langle 0|a_k \left( a_k^\dagger a_{k'} + (2\pi)^3 N_k \delta^3(\vec{k} - \vec{k}') \right) |0\rangle \tag{169}$$

$$+ e^{ikx} \langle 0| \left( a_{k'}^\dagger a_k + (2\pi)^3 N_k \delta^3(\vec{k} - \vec{k}') \right) a_k^\dagger |0\rangle \Big) \tag{170}$$

$$= 0 \tag{171}$$

where we have used that $a_k |0\rangle = 0$ and $\langle 0| a_k^\dagger = 0$. So the expectation value of the field in a one-particle state is zero. That is, if we measure the field value at any point at any time, we will get **on average** 0. This, however, does not mean that the field is not moving, since you can check that

$$\langle 0|a_k (\Phi_I(t, \vec{x})^2 a_k^\dagger|0\rangle \neq 0 \tag{172}$$

Squaring our field adds upwards and downward fluctuations coherently, so we conclude it is moving (oscillating) around 0. We picture then our states as "ripples" on the field, around an average level as shown in fig. 10.

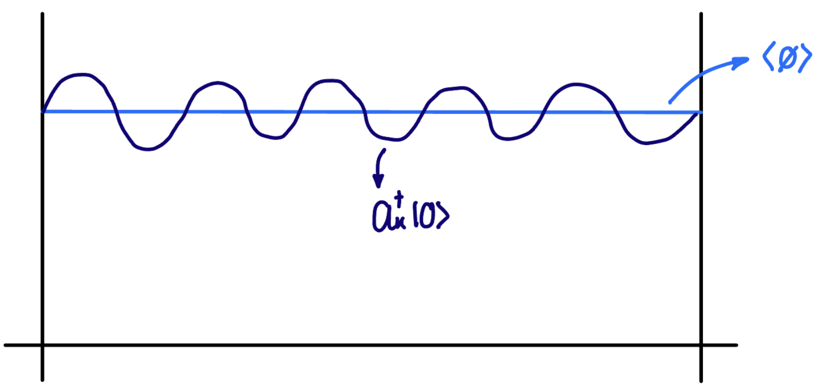



Figure 10: "Ripples" of a field in dark blue, around an average level shown in light blue"

You can check that $\langle 0|(a_k)^n \phi (a_k^\dagger)^n|0\rangle = 0$ for any $n$. States with a well-defined number of particles return 0 average value for the field. We can build $\langle \phi \rangle \neq 0$ with a superposition of different

$n$ states. This allows us to qualify the statement "the light coming from that lamp is made out of photons". Actually, "the light coming from that lamp is a superposition of multiphoton states".

### 3.2.1 Number and Momentum Operator

What we mean by well-defined number of particles can be made explicit

$$\mathcal{N} = \int [d\vec{k}] a_k^\dagger a_k \qquad\qquad |n\rangle = (a_k^\dagger)^n |0\rangle \tag{173}$$

$$\mathcal{N}|n\rangle = n|n\rangle \tag{174}$$

An eigenstate of $\mathcal{N}$ then has a well-defined number of particles & since

$$[\mathcal{N}, \Phi] = 0$$

they cannot be simultaneously $\Phi$ eigenstates. This counting of particles can be used to define **total momentum** $\mathrm{P}^i$

$$\mathrm{P}^i a_k^\dagger |0\rangle = \hbar k^i a_k^\dagger |0\rangle \tag{175}$$

$$\mathrm{P}^i a_{k_1}^\dagger a_{k_1}^\dagger |0\rangle = ? \tag{176}$$

where the RHS of eq. (176) is left for you to work out. The operator that does this job is:

$$\mathrm{P}^i \equiv \int [d\vec{k}] \hbar k^i a_k^\dagger a_k \tag{177}$$

## 3.3 Connection with 'old' QM

Defining a position operator is not as simple. There is no wavefunction in sight to tell us what a localised particle at $\vec{x}$ looks like. The general procedure is to look at the energy density:

$$\langle s|\mathcal{H}(\vec{x})|s\rangle \sim |\text{wavefunction}(\vec{x})|^2$$

but this equation makes no sense in general, since the LHS need not factorise into the square of anything. One can make sense of this for the *non-relativistic limit* and proceed as follows: with our non-relativistic states

$$\left|\vec{k}\right\rangle \equiv \frac{a_k}{\sqrt{N_k}} \left|\vec{k}\right\rangle \tag{178}$$

$$\left\langle \vec{k}' \middle| \vec{k} \right\rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \tag{179}$$

we Fourier transform for a position state

$$|\vec{x}\rangle \equiv \int \frac{d^3k}{(2\pi)^3} \left|\vec{k}\right\rangle \left\langle \vec{k} \middle| \vec{x} \right\rangle = \int [d\vec{k}] e^{-i\vec{k}\cdot\vec{x}} \frac{N_k}{\sqrt{N_k}} a_k^\dagger |0\rangle \tag{180}$$

your position operator then

$$\mathrm{X} \equiv \int d^3y \vec{y} |\vec{y}\rangle \langle \vec{y}| \qquad\qquad \mathrm{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle \tag{181}$$

and your wavefunction

$$|s\rangle = \int d^3y |y\rangle \langle y|s\rangle \equiv \int d^3y \psi(y) |\vec{y}\rangle \tag{182}$$

Note that

$$\langle\vec{x}|\vec{y}\rangle = \int [dk][dk']\sqrt{N_k}\sqrt{N_{k'}}\,\langle 0|a_{k'}a_k^\dagger|0\rangle\, e^{ik'y} \tag{183}$$

$$= \int [dk] N_k e^{ik(y-x)} \tag{184}$$

$$= \delta^3(\vec{x}-\vec{y}) \tag{185}$$

and is only the identity in a 1-particle subspace.

So the usual action of momentum on the wavefunction

$$\mathrm{P}^i \int d^3x\psi(x)\,|\vec{x}\rangle = \hbar\int [dk]k^i a_k^\dagger a_k \int d^3x\psi(x)\int [d\vec{k}']\sqrt{N_k}e^{-i\vec{k}'\cdot\vec{x}}a_{k'}^\dagger\,|0\rangle \tag{186}$$

$$= \hbar\int d^3x\int [d\vec{k}]a_k^\dagger k^i\psi(x)\sqrt{N_k}e^{-i\vec{k}\cdot\vec{x}}\,|0\rangle \tag{187}$$

$$= \hbar\int d^3x\int [d\vec{k}]a_k^\dagger\,|0\rangle\,\sqrt{N_k}i\frac{\partial}{\partial x^i}e^{i\vec{k}\cdot\vec{x}}\psi(x) \tag{188}$$

$$= \int d^3x\left(-i\hbar\frac{\partial}{\partial x^i}\psi(x)\right)\int [d\vec{k}]\sqrt{N_k}a_k^\dagger e^{i\vec{k}\cdot\vec{x}}\,|0\rangle \tag{189}$$

$$= \int d^3x\left(-i\hbar\frac{\partial}{\partial x^i}\psi(x)\right)|\vec{x}\rangle \tag{190}$$

and the commutation relation is realised by

$$[\mathrm{X}^i,\mathrm{P}^i]\int d^3x\psi(x)\,|\vec{x}\rangle = \int d^3x\left(-i\hbar\frac{\partial\psi(x)}{\partial x^i}\right)|\vec{x}\rangle$$

### 3.4 A First Look at Causality [P.S 2.1-2.4, S 12.6, T 2.6.1]

Special relativity taught us that nothing travels faster than $c$. An immediate consequence is causal ordering. Consider an event taking place at $t_1,\vec{x}_1$ and another at $t_2,\vec{x}_2$; the Minkowski product is

$$(x_1-x_2)^\mu\eta_{\mu\nu}(x_1-x_2)^\nu = (t_1-t_2)^2c^2 - (\vec{x}_1-\vec{x}_2)^2 \tag{191}$$

A light signal from event 1 would have covered a distance $c(t_2-t_1)$ by the time of event 2. If $c(t_2-t_1) < |\vec{x}_2-\vec{x}_1|$, light cannot cover the distance in time and the two events are causally disconnected. The opposite case has that light would have reached event 2 and there is possibility of correlation.

Let's define

| | | |
|---|---|---|
| — | $(x_1-x_2)^\mu(x_1-x_2)^\mu < 0$ | Causally disconnected, or space-like |
| — | $(x_1-x_2)^\mu(x_1-x_2)^\mu > 0$ | Causally connected, or time-like |
| — | $(x_1-x_2)^\mu(x_1-x_2)^\mu = 0$ | Causally connected, or light-like |

which make up the light-cone shown in fig. 11. How do we check for the causality of our theory? There should be no correlation between two measurements at causally disconnected space-time points. Which is to say that one measurement should not affect the other; which is to say that the operators commute since they cannot be simultaneously diagonalised.

$$[\mathcal{O}(t_1,\vec{x}_1),\mathcal{O}'(t_2,\vec{x}_2)] = 0 \tag{192}$$

$$\text{for}\quad c^2(t_1-t_2)^2 - (\vec{x}_1-\vec{x}_2)^2 < 0 \tag{193}$$

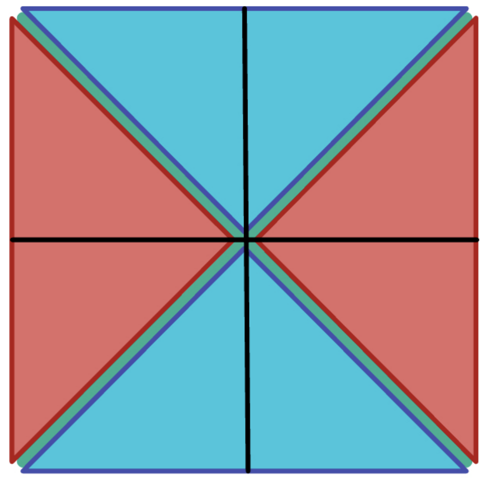


Figure 11: Diagram of the light-cone: blue is time-like, red space-like and green is light-like.

We build our operators out of the field & its derivatives; it can be shown that for causality to hold, the commutator of the field itself should vanish for space-like points.

#### 3.4.1 Example: Complex, Constrained Field

Causality is not guaranteed; take our field $\varphi$

$$\mathcal{L}_S = \varphi^* i\hbar \frac{\partial}{\partial t}\varphi - \frac{\hbar^2 \nabla\varphi^* \nabla\varphi}{2m} \tag{194}$$

which is an attempt at a non-relativistic quantum field theory. As such, this theory does not know about the speed of light, so one wouldn't expect it to be causal. Let's show it is not:

$$[\varphi(x_1), \varphi^\dagger(x_2)] = \int [d\vec{k}][d\vec{k}'] \hbar \sqrt{N_k}\sqrt{N_{k'}} e^{-\frac{iE_k t_1}{\hbar} + i\vec{k}\cdot\vec{x}_1} e^{-\frac{iE_{k'} t_2}{\hbar} + i\vec{k}'\cdot\vec{x}_2} [a_k, a_k^\dagger] \tag{195}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{\frac{i\hbar k^2}{2m}\Delta t - i\vec{k}\cdot\Delta\vec{x}} \quad \text{if } t_2 > t_1 \tag{196}$$

we want to find out if this expression cancels for $c^2\Delta t^2 - |\Delta x|^2 < 0$. Before doing the integral explicitly, one can get a sense by looking at the speed of the waves we are integrating over

$$v \sim \frac{\partial\omega}{\partial|\vec{k}|} = \frac{\partial}{\partial|\vec{k}|}\frac{\hbar^2\vec{k}^2}{\hbar 2m} = \frac{\hbar|\vec{k}|}{m} \tag{197}$$

> Remember for our electromagnetic potential $\Phi$ in lecture 1, $\omega = |\vec{k}|c$.

This grows ever larger for larger $|\vec{k}|$ and for $|\vec{k}| > mc/\hbar$ waves are superluminal. Not a good sign for causality. The final confirmation comes after actually doing the integral, done in the complex place as a Gaussian to yield for $\delta t \neq 0$

$$[\varphi(x_1), \varphi^\dagger(x_2)] = \left(\frac{m}{2\pi\Delta t\hbar}\right)^{3/2} e^{i\phi} \tag{198}$$

This is non-zero in general & in particular if say $\Delta x = nc\Delta t, n > 1$

$$\frac{1}{n^{3/2}}\left(\frac{m}{2\pi\hbar c\Delta x}\right)^{3/2} \neq 0 \tag{199}$$

which does decrease the further away the points are, but we were looking for zero, not small. **So this theory does not respect causality**.

# 4 C: Symmetry [P.S. 2.2, S 2.1, Z I.10, T 1.3]

Here we choose to present different concepts separately; hopefully this will give you a sense of what each of the moving pieces in QFT does. One, however, cannot get too far without encountering symmetry in particle physics; it permeates through pretty much all of it. An overview of the kinds we'll find

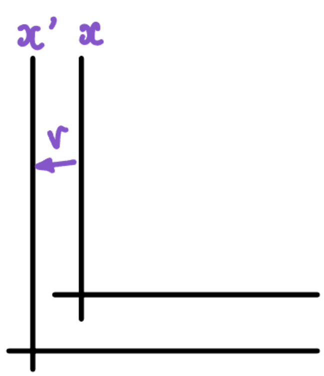
- **Continuous:** Any transformation has others infinitesimally close. E.g. rotations.
- **Discrete:** None of the elements can be obtained from an infinitesimal displacement. E.g. parity.
- **Global:** Independent of space-time. E.g. baryon number.
- **Local (gauge):** Space-time dependent. E.g. $U(1)_{\text{em}}$ diffeomorphisms.
- **Internal:** Acting on fields only.

## 4.1 Boosts & Rotations

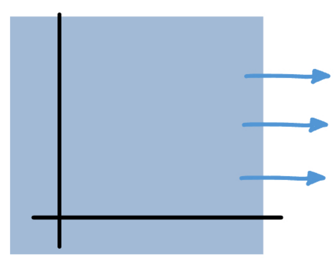
You might know about Lorentz transformations, such as a boost

$$\begin{pmatrix} ct' \\ \vec{x}' \end{pmatrix} = \begin{pmatrix} \gamma 1 & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$$

for $\beta = v/c$ and $\gamma = (1 - v^2/c^2)^{-1/2}$.




(a) passive: a relation between frames

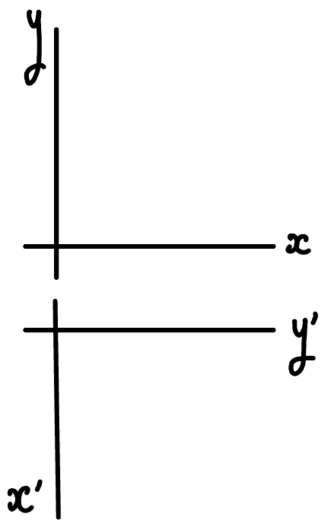


(b) active: act on spacetime "shift fabric"

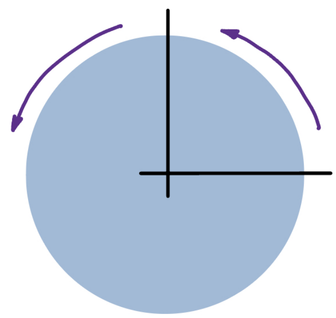
Or a rotation, such as

$$\begin{pmatrix} ct' \\ \vec{x}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$$

such that $x' = -y$ and $y' = x$.




(a) Passive rotation.



(b) Active rotation.

### 4.2 A Group

If we perform a rotation, then another, the resulting action is still a rotation. If we do a rotation, then a boost, out comes an action which is neither of the two; it is a general Lorentz transform. This is the end of this game: performing two Lorentz transforms is still a Lorentz transform. All this talk is formalised in **group theory**. A group $B$ is a set of elements $U_i$ with a composition rule $U_1 \circ U_2$ such that

(i) It contains the identity.

(ii) For each element, it's inverse is in the group.

(iii) Associativity $U_1 \circ (U_2 \circ U_3) = (U_1 \circ U_2) \circ U_3$.

(iv) Closure $U_1 \circ U_2 = U_3$.

When acting on a linear representation $r$:

$$t \to r' = U(\theta) r = e^{i\theta_a T^a} r \tag{200}$$

where $a$ runs up to the dimension of the group, $\theta_a$ are real parameters and $T^a$ are the group generators. By (iv) above, $[T^a, T^b] \propto T^c$.

### 4.3 Lorentz Group

Let's apply it to the Lorentz group. The Lorentz transformation rule is $x'^\mu = \Lambda^\mu{}_\nu x^\mu$ which defines the group. We will consider an infinitesimal boost by small $\vec{\beta}$ and rotation by a small angle $\theta$

$$\text{Boost: } \Lambda^\mu{}_\nu = 1 + \begin{pmatrix} 0 & \vec{\beta} \\ \vec{\beta} & 0 \end{pmatrix} + \mathcal{O}(\beta^2) \tag{201}$$

$$\text{Rotation: } \Lambda^\mu{}_\nu = 1 + \begin{pmatrix} 0 & \vec{0} \\ \vec{0} & \varepsilon_{ijk}\theta^k \end{pmatrix} + \mathcal{O}(\theta^2) \tag{202}$$

which we can combine into a single infinintesimal $\omega^{\mu}{}_{\nu}$

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{203}$$

Using the fact that the Minkowski product is invariant:

$$x^{\mu} x_{\mu} = (x^0)^2 - (\vec{x})^2 = (x')^{\mu}(x')_{\mu} \tag{204}$$
$$\Lambda^{\mu}{}_{\nu} x^{\nu} \eta_{\mu\rho} \Lambda^{\rho}{}_{\kappa} x^{\kappa} = x^{\nu} \eta_{\nu\kappa} x^{\kappa} \tag{205}$$

$$\boxed{\Lambda^{\mu}{}_{\nu} \eta_{\mu\rho} \Lambda^{\rho}{}_{\kappa} = \Lambda^{\mu}{}_{\nu} \Lambda_{\mu\kappa} = \eta_{\nu\kappa}} \tag{206}$$

Then applying Eq. 203

$$\Lambda^{\mu}{}_{\nu} \eta_{\mu\rho} \Lambda^{\rho}{}_{\kappa} \sim \eta_{\nu\kappa} + \omega^{\mu}{}_{\nu} \eta_{\mu\kappa} + \eta_{\nu\rho} \omega^{\rho}{}_{\kappa} = \eta_{\nu\kappa} \tag{207}$$

$$\boxed{\omega_{\kappa\nu} + \omega_{\nu\kappa} = 0} \tag{208}$$

This all agrees;

$$\eta \cdot \begin{pmatrix} 0 & \beta \\ \beta & \varepsilon\theta \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & -\varepsilon\theta \end{pmatrix}$$

is anti-symmetric and it allows us to redo counting: $\omega + \omega^T = 0$ means we have 6 free parameters. This group gets its name from its definition:

$$\begin{array}{ccc} \Lambda^T \eta \Lambda = \eta & \mathcal{O}(1,3) & \text{Disconnected} \\ \text{for } \det\{n\} = 1 & \mathcal{SO}(1,3) & \text{Connected} = \textbf{Lorentz group} \end{array}$$

## 4.4 $U(1)$ Group

A simpler yet ubiquitous group is $U(1)$, defined to be the unitary complex transformations in 1 complex dimension.

$$\begin{array}{ll} \varphi' = U\phi & (\varphi')^{\dagger}(\varphi') = \varphi^{\dagger}\varphi \\ U = e^{i\theta} & \theta \text{ is one parameter, } T = 1 \end{array}$$

If we have another representation of the group, it'll transform the same up to a factor

$$\psi' = e^{iq\theta} \qquad q \text{ is a ratio of } \psi, \phi\ U(1) \text{ charges.}$$

One question in QED is why all ratios are rational, i.e. we have charge quantisation. These are group definitions, **for a group to lead to a symmetry, we say that its transformations leave the action invariant (the same) up to a boundary (i.e. a total derivative) term.**

## 4.5 Lorentz Symmetry

You might have heard that the laws of physics are Lorentz invariant. This does *not* mean things look the same, it means the *only information* we need to relate two frames of reference is $\Lambda^{\mu}{}_{\nu}$ (see the schematic in fig. 14). E.g. we do not have to solve the equations of motions twice.

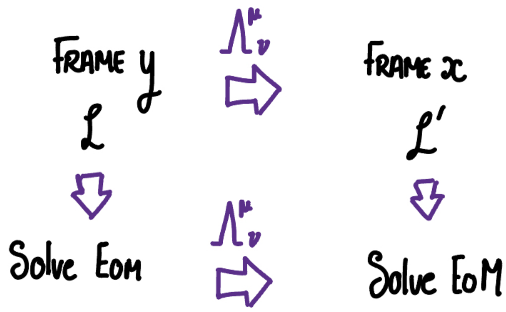



Figure 14: Schematic of transforming between frames using that laws of physics is Lorentz invariant.

Let's try it out. Say $x = \Lambda y$ and $y = \Lambda^{-1}x$, then

$$\phi' = \phi(\Lambda^{-1}x) \tag{209}$$

$$(\partial_\mu \phi)' = \frac{\partial}{\partial x^\mu}\phi(\Lambda^{-1}x) = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\phi(y) = (\Lambda^{-1})^\nu{}_\mu \partial_\nu(\Lambda^{-1}x) = \Lambda_\mu{}^\nu(\partial_\nu\phi) \tag{210}$$

$$(\partial^\mu\phi)' = \Lambda^\mu{}_\nu(\partial^\nu\phi) \tag{211}$$

where the final step in Eq. 210 uses a result that will be derived in the problem sheet.

### 4.5.1 Klein-Gordon Field

Let's see how a transformation in the Lorentz group will affect the Klein-Gordon equation. Recall for a relativistic energy relation, we had that the action $S$ is

$$S = \int dt d^3x \frac{1}{2}\left((\partial_t\phi)^2 - (c\nabla\phi)^2 - \frac{m^2c^4}{\hbar^2}\phi^2\right) \tag{212}$$

$$= \int \frac{d^4x}{c}\frac{c^2}{2}\left(\left(\frac{\partial\phi}{c\partial t}\right)^2 - (\nabla\phi)^2 - \frac{m^2c^2}{\hbar^2}\phi^2\right) \tag{213}$$

$$= c\int d^4x \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - \frac{m^2c^2}{\hbar^2}\phi^2\right) \tag{214}$$

so the Euler Lagrange equation in frame $y$

$$\eta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 \tag{215}$$

If we find a solution $\phi(y)$, would $\phi' = \phi(\Lambda^{-1}x)$ be a solution in frame $x$?

$$\eta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}\phi' + \frac{m^2c^2}{\hbar^2}\phi' = \Lambda_\mu{}^\rho\frac{\partial}{\partial y^\rho}\Lambda^\mu{}_\nu\frac{\partial^\nu\phi}{\partial y} + \frac{m^2c^2}{\hbar^2}\phi \tag{216}$$

$$= \Lambda_\mu{}^\rho\Lambda^\mu{}_\nu\frac{\partial}{\partial y^\rho}\eta^{\kappa\nu}\frac{\partial}{\partial y^\kappa}\phi + \frac{m^2c^2}{\hbar^2}\phi \tag{217}$$

$$= \left(\frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y_\nu} + \frac{m^2c^2}{\hbar^2}\right)\phi \tag{218}$$

i.e. yes! This follows from the action being invariant under Lorentz transformations, as we will also show. For now though, let's talk units.

## 4.6 Natural Units

Two fundamental constants of nature are

$$c = 2.998 \cdot 10^8 \text{ m/s} \tag{219}$$

$$\hbar = 6.582 \cdot 10^{-16} \text{ eVs} \tag{220}$$

They show up all over our formulae, so we will set them to 1 and say we measure speed in units of $c$ and angular momentum in units of $\hbar$.

$$c = 1 \qquad\qquad \hbar = 1 \tag{221}$$

As we outlined, this implies

$$[E] = [\text{mass}] = [\text{length}]^{-1} = [\text{time}]^{-1} = \text{eV} \tag{222}$$

To convert back and forth, then we restore powers of $\hbar$ and $c$. For example if we have an area $A$:

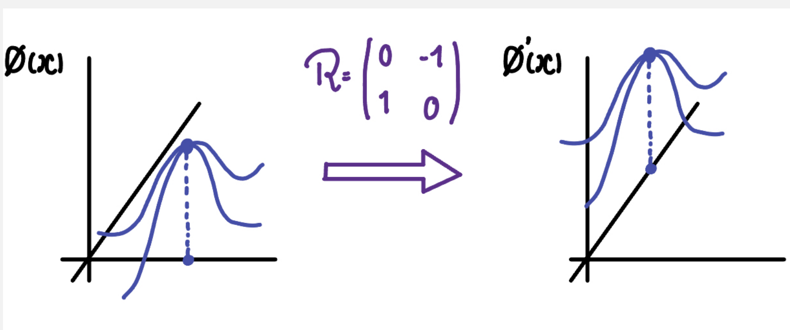
$$A_{\text{N.U.}} = \frac{10^{-20}}{\text{eV}^2} \qquad\qquad A_{\text{S.I.}} = A_{\text{N.U.}} \hbar^p c^q \tag{223}$$

$$[A_{\text{S.I.}}] = m^2 = A_{\text{N.U.}} (\text{eVs})^p (\text{m/s})^q = \text{eV}^{-2} (\text{eVs})^p (\text{m/s})^q \tag{224}$$

where we see that $p = q = 2$ and as such,

$$A_{\text{S.I.}} = \frac{10^{-20}}{\text{eV}^2} (2 \cdot 10^{-7} \text{ m eV})^2 \tag{225}$$

From now on we'll take the view of an active transformation on the fields.

> For example an active field rotation:
>
> 
> 
>
> $$\phi'\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \phi'\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \left(\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right) \tag{226}$$
>
> $$\phi'(x) = \phi(R^{-1}x) \tag{227}$$

For a general transformation

$$\phi'(x^\mu) = \phi\left((\Lambda^{-1})^\mu{}_\nu x^\nu\right) \tag{228}$$

Computationally, the same results follow as before

$$(\partial^\mu \phi)' = \Lambda^\mu{}_\nu (\partial^\nu \phi)((\Lambda^{-1})x) \tag{229}$$

Next, we show that the action is left invariant.

$$S' = \int d^4x \det(\Lambda) \left\{ \frac{1}{2} (\partial_\mu \phi)' (\partial^\mu \phi)' - V(\phi') \right\} \tag{230}$$

$$= \int d^4x \left\{ \frac{1}{2} \Lambda_\mu{}^\nu (\partial_\nu \phi)(\Lambda^{-1}x) \Lambda^\mu{}_\rho (\partial^\rho \phi)(\Lambda^{-1}x) - V(\phi(\Lambda^{-1}x)) \right\} \tag{231}$$

for an infinitesimal Lorentz transformation as in Eq. 203:

$$\delta_\omega S = \int d^4x \left( \frac{1}{2} \partial_\mu \phi(x - \omega x) \partial^\mu \phi(x - \omega x) - V(\phi(x - \omega x) \right) \tag{232}$$

$$= \int d^4x \left( -\omega^\nu{}_\rho x^\rho \partial_\nu \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right\} \right) \tag{233}$$

$$= \int d^4x \left( -\partial_\nu \omega^\nu{}_\rho x^\rho \left\{ \frac{1}{2} (\partial_\mu \phi) \partial^\mu \phi - V(\phi) \right\} \right) \tag{234}$$

So this change is just a total derivative, and so the integral only depends on the boundary. As such the K.G. equation is invariant under $SO(1,3)$. The $U(1)$ case is simpler. For the $U(1)$ transformation $\phi' = e^{i\theta}\phi$ where $\theta$ is small:

$$S' = \int d^4x \varphi^\dagger e^{-i\theta} \partial_t e^{i\theta} \varphi - \frac{\nabla \varphi^\dagger e^{-i\theta} \nabla e^{i\theta} \varphi}{2m} \tag{235}$$

$$\delta_\theta S = \int d^4x (i\theta - i\theta) \mathcal{L} = 0 \tag{236}$$

## 4.7 Noether's Theorem

One can further expect the presence of symmetries to obtain conservation laws. Here's how, take the infinitesimal transformation of the field $\phi$ to be $\delta_\theta \phi$. We say the transformation is a symmetry if the Lagrangian changes by a total derivative

$$\delta \mathcal{L} = \partial_\mu F^\mu \tag{237}$$

i.e. the action is left invariant

$$\delta_\theta S = \int d^4x \partial_\mu F^\mu_a \theta^a \tag{238}$$

$$= \int d^4x \left( \partial_\mu (\delta_\theta \phi) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + \delta_\theta \phi \frac{\partial \mathcal{L}}{\partial \phi} \right) \tag{239}$$

$$= \int d^4x (\delta_\theta \phi)(\text{–EoM–}) + \int d^4x \partial_\mu \left( \delta_\theta \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \tag{240}$$

which we can rearrange to be

$$\partial_\mu \left( \delta_\theta \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - F^\mu_a \theta^a \right) \equiv J^\mu_a \theta^a = 0 \tag{241}$$

where $a = 1, ..., \dim(G)$ where $G$ is the transformation's group. And $J_a^\mu$ is known as a Noether current, which is conserved over a volume $V$:

$$\int_V d^3x \partial_\mu J^\mu = \frac{d}{dt} \int_V d^3x J_a^0 + \int_A d\vec{\Sigma} \cdot \vec{J} = 0 \tag{242}$$

The second term measures the flow out of the surface of area $A$ (see fig. 15).

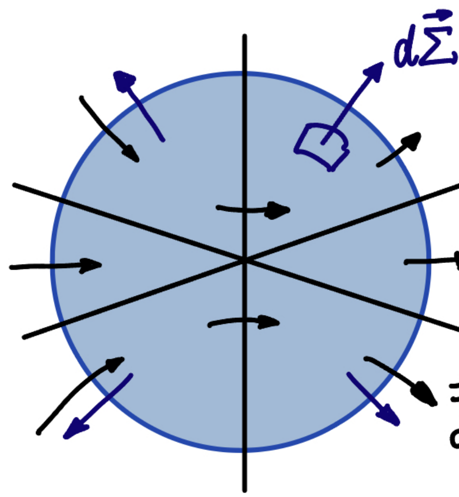



Figure 15: A surface, area $A$ with surface element $d\vec{\Sigma}$, and a current $\vec{J}$ flowing from it.

If we take the surface to be large enough, we assume the fields in $\vec{J}$ die off and $\vec{J} \to 0$ fast enough to cancel the integral such that

$$\mathcal{Q}_a \equiv \int_V d^3x J_a^0 \qquad\qquad \frac{d}{dt}\mathcal{Q}_a = 0 \tag{243}$$

we have obtained a **conserved charge!** Next is a tour of conserved charges and currents, some of which are fundamental.

## 4.8 Translations

We skipped this one but fundamental physics does not care about where you set the origin of space-time. Or, in terms of a transformation

$$x \to x' = x - \epsilon \tag{244}$$

In face, in combination with the Lorentz group, they give us the Poincare group

$$x' = \Lambda x - \epsilon \tag{245}$$

In our active interpretation, our field transforms as

$$\phi' = \phi(x + \epsilon) \qquad\qquad (x' = x - \epsilon) \tag{246}$$

$$\delta_\epsilon = \epsilon^\mu \partial_\mu \phi \tag{247}$$

These don't affect our derivatives as $\partial x'/\partial x = 1$ as spacetime has only been shifted. Our new action is

$$S' = \int d^4x \mathcal{L}\left(\phi(x+\epsilon), \partial_\mu \phi(x+\epsilon)\right) \tag{248}$$

which gives a variation

$$\delta S = \int d^4x \left[(\epsilon^\nu \partial_\nu \partial_\mu \phi)\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + (\epsilon^\nu \partial_\nu \phi)\frac{\partial \mathcal{L}}{\partial \phi}\right] \tag{249}$$

$$= \int d^4x \left[\epsilon^\nu \partial_\nu \mathcal{L}\right] \tag{250}$$

$$= \epsilon^\nu \int d^4x \left(\partial_\nu \eta^\mu{}_\nu \mathcal{L}\right) \tag{251}$$

such that $F^\mu{}_{(\nu)} = \eta^\mu{}_\nu \mathcal{L}$, one for each $\nu$. I.e. we have four conserved currents:

$$\epsilon^\nu J^\mu{}_{(\nu)} = \left(\delta_\epsilon \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \epsilon^\nu \eta^\mu{}_\nu \mathcal{L}\right) \tag{252}$$

$$J^\mu{}_\nu \equiv T^\mu{}_\nu = \left(\partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \eta^\mu{}_\nu \mathcal{L}\right) \tag{253}$$

This is none other than the Stress-Energy tensor. For our **real K.G. Field**

$$T^m u\ {}_\nu = \partial_\nu \phi \partial^\mu \phi - \eta^\mu_\nu \left(\frac{(\partial \phi)^2}{2} - \frac{m^2\phi^2}{2}\right) \tag{254}$$

Do you recognise the ${}^0{}_0$ component? Yep, it's the Hamiltonian

$$\dot{\phi}^2 - \frac{1}{2}\left(\dot{\phi}^2 - (\nabla \phi)^2 - m^2\phi^2\right) \tag{255}$$

Indeed, $\epsilon^0$ is a time translation that leads to energy conservation.

$$H = \int d^3x T^0{}_0 \tag{256}$$

If you remember, a space shift does lead to total momentum conservation

$$P^i = \int d^3x T^{0i} \tag{257}$$

which for K.G. is

$$P^i = \int d^3x \dot{\phi} \partial^i \phi = \int d^3x \dot{\phi}\left(-\frac{\partial \phi}{\partial x^i}\right) \tag{258}$$

We will connect this to the previous in the exercises.

### 4.8.1 Alternative Derivation

For general space-time we substitute $\eta_{\mu\nu} \to \partial_{\mu\nu}(x)$ and the action is

$$\int d^4x \sqrt{|g|} \mathcal{L}(\phi, \partial_\mu \phi, \eta \to g) \tag{259}$$

where $g_{\mu\nu}$, the metric, is also the graviton field. We can define $T^{\mu\nu}$ as the source, i.e. the linear coupling of $g_{\mu\nu}$.

$$T^{\mu\nu} \equiv -\left(\frac{2}{\sqrt{g}}\frac{\partial}{\partial g_{\mu\nu}}\sqrt{g}\mathcal{L}(\eta \to g)\right)_{g=\eta} \tag{260}$$

You can check this explicitly using

$$\frac{\partial}{\partial g_{\mu\nu}} g_{\alpha\beta}g^{\beta\gamma} = 0 \Rightarrow \frac{\partial g^{\beta\gamma}}{\partial g_{\mu\nu}} = -g^{\mu\beta}g^{\nu\gamma} \tag{261}$$

$$|g| = |det(g_{\mu\nu})| \tag{262}$$

$$\frac{\partial}{\partial g_{\mu\nu}} = g^{\mu\nu}|g| \tag{263}$$

## 4.9 U(1) Current

The $U(1)$ symmetry has no $F$ term so given $\delta\phi_\theta = i\theta\phi$, $\delta_\theta\phi^* = -i\theta\phi^*$.

$$J^\mu = i\left(\phi\frac{\partial\mathcal{L}}{\partial(\partial_m u\phi)} - \phi^*\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\right) \tag{264}$$

Examples are

$$\mathcal{L}_{\text{K.G.}} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \tag{265}$$

$$\mathcal{L}_{\text{S}} = i\varphi^*\partial_t\varphi - \frac{\nabla\varphi^*\nabla\varphi}{2m} \tag{266}$$

with currents

$$J^\mu_{\text{K.G.}} = i\left(\partial^\mu\phi^*\phi - \phi^*\partial^\mu\phi\right) \tag{267}$$

$$J^\mu_{\text{S}} = i\left(i\varphi^*\varphi, -\left(\frac{-(\vec{\nabla}\varphi^*)\varphi + \varphi^*(\vec{\nabla}\varphi)}{2m}\right)\right) \tag{268}$$

## 4.10 Lorentz Symmetry

We have that

$$\delta_\omega\phi = -\omega^\mu{}_\nu x^\nu\partial_\mu\phi \tag{269}$$

$$= \omega^{\alpha\beta}(-x_\beta\partial_\alpha\phi) \tag{270}$$

$$= \omega^{\alpha\beta} \tag{271}$$

$$= \omega^{\alpha\beta}\left(-\frac{x_{[\beta}\partial_{\alpha]}\phi}{2}\right) \tag{272}$$

where the square brackets on the final line mean the indices are antisymmetrised (i.e. $x_{[\beta}\partial_{\alpha]}\phi = x_\beta\partial_\alpha\phi - \partial_\alpha x_\beta\phi$. And

$$\delta_\omega S = \int d^4x\partial_\mu\left(-\omega^\mu{}_\nu x^\nu\partial_\mu\mathcal{L}\right) \tag{273}$$

$$= \int d^4x\omega^{\alpha\beta}\left(-x_\beta\partial_\alpha\mathcal{L}\right) \tag{274}$$

$$= \int d^4x\omega^{\alpha\beta}\left(-\frac{x_{[\beta}\partial_{\alpha]}\mathcal{L}}{2}\right) \tag{275}$$

from which our six currents follow as

$$J^{\mu}_{(\alpha\beta)} = \frac{2}{2}\left(-x_{[\beta}\partial_{\alpha]}\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} + x_{[\beta}\partial_{\alpha]}\mathcal{L}\right) \tag{276}$$

$$= \left(x_\alpha T^{\mu}{}_{\beta} - x_\beta T^{\mu}{}_{\alpha}\right) \tag{277}$$

These are not unfamiliar for $\alpha, \beta = 1, 2, 3$:

$$J^{0(ij)} = x^i T^{0j} - x^k T^{0i} \qquad \text{Rotations} \tag{278}$$

$$J^{0(0i)} = tT^{0i} - x^i\mathcal{H} \qquad \text{Other translations} \tag{279}$$

To clarify, let's write

$$\frac{d}{dt}\left(\int d^3x(tT^{0i}) - \int d^3x x^i\mathcal{H}(x)\right) = 0$$

$$tP^i = \int d^3x x^i\mathcal{H}(x) + \mathcal{C}$$

which tells us the centre of mass (energy) moves inertially ($dP^i/dt = 0$). While on the topic,

$$\int \frac{d^3k}{(2\pi)^3 2E_k} = \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\Theta(k^0) \tag{280}$$

is the true (and only) Lorentz invariant measure, as you will show on the problem sheet.

## 4.11 Causality: Lorentz Invariant Theories [P.S. 2.1&2.4, S 12.6, T 2.1.6]

We can now revisit causality for Lorentz Invariant theories. Take real K.G. theory, the field itself does not commute for any $x^\mu, y^\mu$

$$[\phi(x), \phi(y)] = \int [dk][dk']\sqrt{\frac{N_k}{2E_k}}\sqrt{\frac{N_{k'}}{2E_{k'}}}[a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}, a_k e^{-ik\cdot y} + a_k^\dagger e^{ik\cdot y}] \tag{281}$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k}\left(e^{-i(x-y)_\mu k^\mu} - e^{-i(y-x)_\nu k^\nu}\right) \tag{282}$$

Where normalisation has dropped out to give a Lorentz invariant result and $k^\mu = (E_k, \vec{k})$. We can use now, for space-like points, laying $\vec{x} - \vec{y} = (0, 0, \Delta d)$

$$\begin{pmatrix}\Delta x^0\\ 0\\ 0\\ \Delta x\end{pmatrix} = \begin{pmatrix}\gamma & 0 & 0 & \gamma\beta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \gamma\beta & 0 & 0 & \gamma\end{pmatrix}\begin{pmatrix}0\\ 0\\ 0\\ \Delta d\end{pmatrix} \tag{283}$$

$$\Delta x^0 = \gamma\beta\Delta d \qquad \Delta x = \gamma\Delta d \tag{284}$$

As such, $\Delta x^0 < \Delta x$ only for causally disconnected (space-like) points. The first term of eq. 282 then reads in this frame

$$\int \frac{d^3k}{(2\pi)^3 2E_k} e^{-i(k\circ\Lambda)_\mu \circ \Delta d^\mu} = \int \frac{d^3k'}{(2\pi)^3 2E_{k'}} e^{-ik'\cdot\Delta d} \tag{285}$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k} e^{ik\Delta d\cos\theta} \tag{286}$$

$$= \int \frac{d\phi \sin\theta d\theta |\vec{k}|^2 d|\vec{k}|}{(2\pi)^3 2E_k} e^{ik\Delta d\cos\theta} \tag{287}$$

$$= \int \frac{|\vec{k}|d|\vec{k}|}{(2\pi)^3 E_k} \frac{\sin\Big(\Delta d\,|\vec{k}|\Big)}{\Delta d} \tag{288}$$

$$= \frac{m^2}{(2\pi)^2\Delta d} K_1(\Delta d\, m) \tag{289}$$

where $K_1$ is a modified Bessel function of the first kind.

This term has a faster decrease than our non relativistic example.

$$\rightarrow e^{-m\Delta d} \qquad\qquad m\Delta d >> 1 \tag{290}$$

But we want zero for the commutator in eq. 282 for causality to be preserved. The way we have written it, the second term is obtained by taking $\Delta d \rightarrow -\Delta d$

$$\int \frac{d^3k}{(2\pi)^3 2E_k} e^{ik\cdot\Delta d} = \int \frac{|\vec{k}|d|\vec{k}|}{(2\pi)^3 E_k} \frac{\sin\Big(-\Delta d\,|\vec{k}|\Big)}{-\Delta d}. \tag{291}$$

We see then, before even carrying out the integral, that:

$$[\phi(x), \phi(y)] = \int_0^\infty \frac{kdk}{(2\pi)^3 e_k} \frac{\sin(\Delta d\; k)}{\Delta d} - \int_0^\infty \frac{\sin(-\Delta d\; k)}{-\Delta d} \tag{292}$$

$$= 0 \tag{293}$$

for $(x-y)^2 < 0$ and using $\Delta x = \gamma\Delta d$ and $\Delta x^0 = \gamma\beta\Delta d$ as in Eq. 284. Causality is thus preserved, though it should be said, L.I. theories don't have the exclusive priviledge of causality. The same exercise for complex K.G. theories goes the same, now with $[a, a^\dagger]$, $[b, b^\dagger]$ terms cancelling out which is why some say the antiparticle restores causality.

# 5 D: Interactions [P.S. 4.5&4.6, S 5.1&6.3, Z I.7, T 3.1&3.2&3.6]

At the heart of it, interactions would be solved if we found the eigenbasis for the full Hamiltonian. Implicit in this view is the assumption that we have a pretty good idea of what the particles are, with $\langle 0|\,\phi\,|\vec{p}\rangle = 0$, and the dispersion relation being $p^2 = m^2$. This however might itself change, and particles move somewhere else; e.g. in QCD we formulate the theory with quark fields $q$, yet particles at low energy are instead in states $\langle 0|\,\bar{q}\gamma_5 q\,|\text{pion}\rangle$. The way to "look" for particles translates then to poles in correlation functions (which is beyond this course).

So we will assume a perturbative expansion holds & gives us small corrections, as is the case in QED.

## 5.1 Dyson's Formula

Time ordering & Dyson's formula are not restricted to QFT, in face one would encounter them in Q.M. with explicit time-dependence. The problem at hand is to evolve in time our operators $(\theta)$ or states $|s\rangle$. Actually if we introduce the evolution operator $U$, it doesn't actually matter which we evolve:

$$\mathcal{O}(t,\vec{x}) = U^\dagger(t,t_i)\mathcal{O}(t_i,\vec{x})U(t,t_i) \tag{294}$$
$$|\psi(t)\rangle = U(t,t_i)\,|\psi(t_i)\rangle \tag{295}$$

For $U$ captures the time-dependence

$$i\partial_t U = HU = (H_0 + U_I)U \tag{296}$$

where we have separated $H$ in two; it is convenient to do the same for $U$ only given our $t=0, \phi(\vec{x})$ which we do as

$$U(t,t_i) \equiv U_0(t,0)U_I(t,t_i)U_0^\dagger(t_i,0) \tag{297}$$
$$\Big[\,|\psi_I(t)\rangle \equiv U_0^\dagger(t,0)\,|\psi(t)\rangle\,\Big] \tag{298}$$

where it follows that $U(t,t) = U_I(t,t) = \mathtt{1}$ and

$$(i\partial_t U)U_0(t_i) = \cancel{i\partial_t U_0(t) U_I(t,t_i)} + iU_0(t)\partial_t U_I(t,t_i) \tag{299}$$
$$= (\cancel{H_0} + H_I)U_0(t)U_I(t,t_i) \tag{300}$$

where the two scored-out terms cancel. Rewriting, we have:

$$i\partial_t U_I(t,t_i) = U_0(t,0)^\dagger H_I(\phi(\vec{x}))U_0(t,0)U_I(t,t_i) \tag{301}$$
$$i\partial_t U_I = H_I U_I \equiv H_i(t)U_I \tag{302}$$

where we have used that the Hamiltonian is a function of the field $\phi(\vec{x})$, and $U_0(t)$ puts the time-dependence given by the tree Hamiltonian to give us the interacting field. The derivative of $U_I$ is therefore the free-Hamiltonian evolved interacting Hamiltonian; that is, for example, $\mathcal{H}_I = \frac{\lambda}{4!}\phi^4$.

$$U_0^\dagger \int d^3x \frac{\lambda}{4!}\phi^4(\vec{x})U_0 = \int d^3x \frac{\lambda}{4!}[U_0^\dagger \phi U_0]^4 \tag{303}$$
$$= \int d^3x \frac{\lambda}{4!}\phi_I(t,\vec{x}) \tag{304}$$

then we can solve iteratively $U_I = \sum_n U_I^{(n)}$:

$$\sum_n i\partial_t U_I^{(n)} = \sum_n H_I(t) U_I^{(n)} \tag{305}$$

which, solving termwise, gives

$$\begin{aligned}
i\partial_t U_I^{(0)} &= 0 & i\partial_t U_I^{(0)} &= 1 \quad (U(t,t) = 1) \\
i\partial_t U_I^{(1)} &= H_I(t) & i\partial_t U_I^{(1)} &= -i\int_{t_i}^{t} dt' H_I(t') \\
i\partial_t U_I^{(2)} &= H_I(t)\left(-i\int_{t_i}^{t} dt' H_I(t')\right) & i\partial_t U_I^{(2)} &= (-i)^2 \int_{t_i}^{t} dt'' H_I(t'') \int_{t_i}^{t''} dt' H_I(t)
\end{aligned}$$

A note on integration, we can perform the time integrals in the manner shown in fig. 16

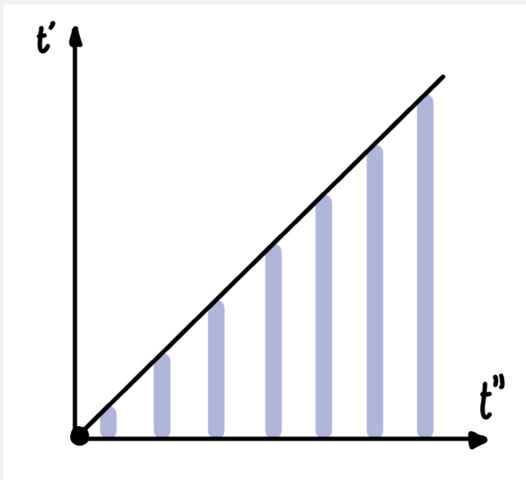



Figure 16: Rearranging the integrand to solve for the time-dependence of the Hamiltonian, eq. (305).

And the solution for general $n$ is:

$$U_I^{(n)} = (-i)^m \int_{t_i}^{t} dt^{(n)} H_I(t^{(n)}) \int_{t_i}^{t^{(n))}} dt^{(n-1)} ... \int_{t_i}^{t''} dt' H_I(t') \tag{306}$$

but it's a little cumbersome to carry around, so we introduce time-ordering $T$

$$\begin{aligned}
T(A(t), B(t')) &= \Theta(t-t')A(t)B(t') + \Theta(t'-t)B(t')A(t) \\
T(A_t, B_{t'}, C_{t''}) &= \Theta_{t-t'}\Theta_{t'-t''}A_t B_{t'} Ct'' + \Theta_{t-t''}\Theta_{t''-t'}A_t Ct'' B_{t'} \\
&+ \Theta_{t'-t}\Theta_{t-t''}B_{t'}A_t Ct'' + \Theta_{t'-t''}\Theta_{t''-t}B_{t'}Ct'' A_t \\
&+ \Theta_{t''-t}\Theta_{t-t'}Ct'' A_t B_{t'} + \Theta_{t''-t'}\Theta_{t'-t}C_{t''}B_{t'}A_t
\end{aligned}$$

e.c.t. For us, all operators are the same and we integrate over all times so

$$T\left\{\int_{t_i}^{t} dt' \int_{t_i}^{t} dt'' H_I(t') H_I^{t''} H_I(t) H_I(t'')\right\} \tag{307}$$

$$= \int dt' dt'' \left(\Theta_{t'-t''} H_I(t') H_I(t'') + \Theta_{t''-t'} H_I(t'') H_I(t')\right) \tag{308}$$

$$= 2\int_{t_i}^{t} dt'' H_I(t'') \int_{t_i}^{t''} dt' H_I(t') \tag{309}$$

where we have used that

$$\int_{t_i}^{t} dt' \Theta_{t''-t'} = \int_{t_i}^{t''} dt'$$

You can show that (Exercise: do it!)

$$T\left\{\prod_{k}^{n} \int dt^{(k)} H_I(t^{(k)})\right\} = n! \int_{t_i}^{t} dt^{(n)} H_I(t^{(n)} ... \int_{t_i}^{t''} dt' H_I(t') \tag{310}$$

which takes us to **Dyson's formula**:

$$\boxed{U_I(t, t_i) = T \exp\left(-i \int_{t_i}^{t} H_I(t') dt'\right)} \tag{311}$$

It is quite a formal achievement which lets us make the time-evolved operator or state of our choice. However, in practice we cannot find an all orders explicit form except for the simplest of cases so we expand on $H_I$.

## 5.2 Scattering and Tree Level Predictions

The formalism of QFT is like a huge machine which, for full rigour and generality, takes a long time to get started and operational. However, if we accept a full approximation as good enough we can short-cut to results. Cause what are we looking for in the end? Predictions to compare with experiment, and the type of experiment we most often make use of in HEP is scattering.

### 5.2.1 What We Measure

It is refreshing to leave the stuffy world of theory for a look at experimental data. The setup is shown in fig. 17

### 5.2.2 Lifetime

One straightforward measure is the lifetime of a particle. A bunch of $N(t)$ unstable particles will decrease in number as

$$\frac{dN}{dt} = -\Gamma N \tag{312}$$

So any one of these has a probability to decay, on a small interval $\Delta t$.

$$\text{Prob} = \frac{-\frac{dN}{dt}\Delta t}{N} = \Gamma \Delta t \tag{313}$$

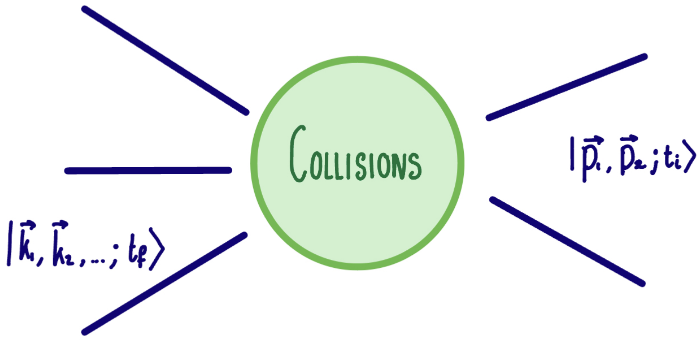



Figure 17: A collision schematic of particles $\vec{p}_1, \vec{p}_2$ colliding at some early time $t_i$, with a final set of particles $\vec{k}_1, \vec{k}_2, ...$ at some late time $t_f$. Note the outgoing state is on the left and incoming on the right, as in e.g. a matrix element calculation.

which we measure experimentally, often quoting the inverse of $\Gamma$:

$$\tau = \Gamma^{-1} \equiv \text{lifetime} \tag{314}$$

$$N(t) = N(t_0)e^{-(t-t_0)/\tau}. \tag{315}$$

For example for muons $\tau_\mu$ $2 \times 10^{-6}$ s, for pions $\tau_\pi$ $3 \times 10^{-8}$ s and for the Higgs $\tau_h \sim 2 \times 10^{-22}$ s.

### 5.2.3 Cross-Section

Picture (by looking at fig. 18) a set of particles directed at another set of target particles.

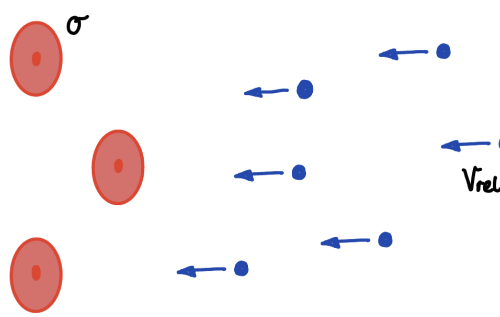



Figure 18: A set of (blue) particles with velocity $v_{\text{rel}}$, relative to some stationary particles (red). The cross-section of the red particles, $\sigma$, is shown as a partially transparent red circle.

We define the cross-section $\sigma$ as the area around a particle within which incoming particles will interact. Then the number of scattering events $N_s$ in a time $\Delta t$ is

$$N_s = \text{Flux} \times N_t \times \sigma \times \Delta t \tag{316}$$

where $N_t$ is the number of target particles, and the flux of the incoming particles is the number of beam particles per unit time per unit area. Any experimentalist with enough data can tell you the cross section

$$\sigma = \frac{N_s}{N_t \times \text{flux} \times \Delta t} \equiv N_s \left[\int dt L_{\text{inst}}\right]^{-1} \tag{317}$$

where we have defined the instantaneous liminosity $L_{\text{inst}}$, which is a beam property of our colliders. For example at the LHC $L_{\text{inst}} \sim \frac{10^{34}}{\text{cm}^2\text{s}}$. Back to our setup, the probability of an incoming particle scattering is given by

$$\text{Prob} = \frac{\text{flux} \times N_t \times \sigma \times \Delta t}{\text{Flux} \times A \times \Delta t} = n_t \times v_{\text{rel}} \times \Delta t \times \sigma \tag{318}$$

with $n_t$ the number density of target particles, i.e. the number of targets per unit volume. Now let's cut it down to the volume that contains on average 1 target particle in a volume $\Delta V$: $n_t \equiv \frac{1}{\Delta V}$. We have then:

$$\text{Prob} = \frac{\Delta t \times v_{\text{rel}} \times \sigma}{\Delta V} \tag{319}$$

and we can make contact with a 1 on 1 scattering fig. 19

Figure 19: One on one scattering process.

which we can connect with the theory expression. We introduce here, with later justification, that for scattering of two incoming particles $k, k'$ into a set of outgoing particles $k_f$ over a time $\Delta t$ is:

$$\langle \{k_f\}, \Delta t | \, U(\Delta t) \left| \vec{k}, \vec{k}' \right\rangle \equiv -i(2\pi)^4 \delta^4(k^\mu k'^\mu - \sum_i k^\mu_{f,i})\mathcal{M} \tag{320}$$

where $\{k_f\}$ stands for the final state and the Dirac delta function follows from energy and momentum conservation, and $\delta^4(k^\mu + k'^\mu - \sum_i k^\mu_{f,i}) = \delta(E_k + E'_k - \sum_i E_{f,i})\delta^3(\vec{k} + \vec{k}' - \sum_i \vec{k}_{f,i})$ and $\mathcal{M}$ is known as the scattering matrix element which we will define later.

We should, however, remember that our states are not normalised. In particular

$$\langle k_1 | k_2 \rangle = 2E_k (2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2) \tag{321}$$

$$= 2E_k \int d^3x e^{i(\vec{k}_1 - \vec{k}_2)\cdot\vec{x}} \tag{322}$$

$$\xrightarrow[k_1 \to k_2]{} 2E_k \Delta V \tag{323}$$

then we need to divide the scattering probability by this factor to normalise

$$\mathrm{Prob} = \sum_f \frac{|\langle\{k_f\}, \Delta t|\, U(\Delta t) \left|\vec{k}, \vec{k}'\right\rangle|^2}{\langle k|k\rangle \langle k'|k'\rangle} \tag{324}$$

$$= \sum_f \frac{|(2\pi)^4\delta^4(k^\mu + k'^\mu - k_f)\mathcal{M}|^2}{2E_k 2E_{k'}(\Delta V)^2} \tag{325}$$

with $k_f = \sum_i k_{f,i}$. Next, we should interpret a Dirac delta squared, which we do by taking the large space-time limit.

$$\int_{-\Delta t/2}^{\Delta t/2} dt e^{it\Delta k^0} = \frac{2}{i\Delta k^0} \sin\left(\frac{\Delta t \Delta k^0}{2}\right) \xrightarrow[\Delta t\to\infty]{} (2\pi)\delta(\Delta k^0) \tag{326}$$

$$\left|\int_{-\Delta t/2}^{\Delta t/2} dt e^{it\Delta k^0}\right|^2 = \left|\frac{2}{i\Delta k^0}\right|^2 \sin^2\left(\frac{\Delta t \Delta k^0}{2}\right) \xrightarrow[\Delta t\to\infty]{} (2\pi)\Delta t\delta(\Delta k^0) \tag{327}$$

which leaves us with

$$\mathrm{Prob} = \sum_f \frac{(2\pi)^4\delta^4(k + k' - k_f)|\mathcal{M}|^2\Delta t\Delta V}{2E_k 2E_{k'}(\Delta V)^2} \tag{328}$$

The sum over final states we obtain from the finite volume as

$$n = \frac{k\cdot L}{2\pi} \qquad\qquad \sum_{n_x n_y n_z} \to \left(\frac{Ldk}{2\pi}\right)^3 \tag{329}$$

$$\frac{1}{\langle k|k\rangle} \Rightarrow \int \frac{\cancel{\Delta V} d^3k}{(2\pi)^3} \frac{1}{2E_k \cancel{(\Delta V)}} \tag{330}$$

We are now in a position to compare with our experimental data:

$$\mathrm{Prob} = \frac{\Delta t \times \sigma \times v_{\mathrm{rel}}}{\Delta V} \tag{331}$$

$$= \sum_f \frac{(2\pi)^4\delta^4(k + k' - k_f)\Delta t\cancel{\Delta V}|\mathcal{M}|^2}{(\Delta V)^{\cancel{2}} 2E_k 2E_{k'}} \tag{332}$$

$$\boxed{\begin{aligned} \sigma &= \frac{1}{2E_k 2E_{k'} v_{\mathrm{rel}}} \sum_f (2\pi)^4\delta^4(k + k' - k_f)|\mathcal{M}|^2 && (333)\\ &= \frac{1}{2E_k 2E_{k'} v_{\mathrm{rel}}} \prod_i \int \frac{d^3k_i}{(2\pi)^3 2E_{k_i}} (2\pi)^4\delta^4(k + k' - k_f)|\mathcal{M}|^2 && (334)\end{aligned}}$$

All factors of time and volume drop, as they should, to give us an expression in terms of Lorentz invariant integrals & matrix element squared. The calculation of the invariant matrix element would then give a prediction for a cross-section to compare with those we measure at the LHC or others.

### 5.2.4 What We Predict We Should Measure

The scattering is mediated by interactions & quite generally we cannot solve exactly for anything other than our tree Hamiltonian. As physicists do when faced with this problem, we expand on *we hope* some small corrections. This is our $H_I$, and formally evolution is solved by Dyson's equation. Further, though, we will make a number of assumptions and approximations:

- We will approximate in and out states, a.k.a. asymptotic states, by tree Hamiltonian eigenstates. This is fine once we renormalise (more on this later).
- Asymptotic states are separated & non-interacting: i.e. they are distant wave-packets. We take the limit of well-defined momenta & approximate them by plane waves. This is also done in e.g. non-relativistic QM.
- We implicitly assumed we knew where the particles "are", i.e. $\langle 0|\,\phi\,|\vec{p}\rangle \neq 0$. This is so for the free theory & stays for weakly coupled theories. But it can change for strongly coupled theories.
- In weakly coupled theories, bound states might show up (e.g. hydrogen atom) which are possible asymptotic states which we ignore for now.

With our first approximation, let us now build our in and out states as in fig. 20.

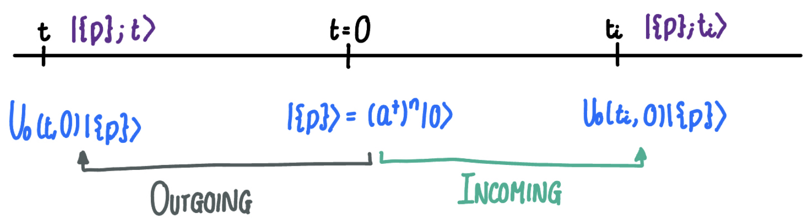



Figure 20: Building our in and out states.

So we define

$$\begin{aligned}\langle\{k\},t|\,U(t,t_i)\,|\{p\},t_i\rangle &= \langle\{k\}|\,U_0^\dagger(t,0)U(t,t_i)U_0(t_i,0)\,|\{p\}\rangle\\ &= \langle\{k\}|\,U_I(t,t_i)\,|\{p\}\rangle\\ &= \langle\{k\}|\,Te^{-i\int dt' H_I}\,|\{p\}\rangle\end{aligned}$$

where we have used eq. (297) and take the limit for $t\to\infty, t'\to-\infty$, then **at tree level:**

$$\langle\{k\}|\,S^{\text{tree}}\,|\{p\}\rangle \equiv \langle\{k\}|\,U_I(\infty,-\infty)\,|\{p\}\rangle \tag{335}$$

$$\equiv \langle\{k\}|\,Te^{-i\int_{-\infty}^{\infty}dt' H_I(t')}\,|\{p\}\rangle \tag{336}$$

This will be our working expression at tree level, with a generalisation that will come in a few lectures.

### 5.2.5 A Lifetime

So let's go ahead and compute something we can measure! Take a real K.G. field $\varphi$ of mass $M$ and complex K.G. field $\phi$ of mass $m$. Then introduce an interaction between them:

$$\mathcal{L}_I = \kappa\varphi\phi^*\phi \qquad\qquad \mathcal{H}_I = -\mathcal{L}_I \tag{337}$$

Provided $M > 2m$ this will mediate decay since the matrix element

$$\langle 0|\hat{a}_{k'}\hat{b}_k S^{\text{tree}} a_p^\dagger|0\rangle \neq 0 \tag{338}$$

where $\hat{a}, \hat{b}^\dagger \leftrightarrow \phi$ and $a \leftrightarrow \varphi$. Let's compute this:

$$\langle 0|\hat{a}_{k'}\hat{b}_k T e^{-i\int H_I(t')dt'} a_p^\dagger|0\rangle = \langle 0|\hat{a}_{k'}\hat{b}_k i\int d^4x\kappa\varphi_I\phi_I^*\phi_I a_p^\dagger|0\rangle \tag{339}$$

Now if we let the $\hat{a}, \hat{b}$ go all the right we get 0, and the same goes if we let $a^\dagger$ go all the way to right left. The non-vanishing piece is then

$$\langle 0|\, \hat{a}_{k'}\hat{b}_k i\int d^4x\int[dq_1]\left(ae^{-iq_1\cdot x} + a^\dagger e^{iq_1\cdot x}\right) \tag{340}$$

$$\cdot\int[dq_2]\left(\hat{a}^\dagger e^{iq_2\cdot x} + \hat{b}e^{-iq_2\cdot x}\right) \tag{341}$$

$$\cdot\int[dq_3]\left(\hat{a}e^{-iq_3\cdot x} + \hat{b}^\dagger e^{iq_3\cdot x}\right) a_p^\dagger\,|0\rangle \tag{342}$$

Note that $[\hat{a}, a^\dagger] = 0$.

$$i\,\langle 0|0\rangle\int d^4x\int[dq_1][dq_2][dq_3]e^{-iq_1\cdot x}[a_{q_1}, a_p^\dagger]e^{iq_2\cdot x}[\hat{a}_{k'}, \hat{a}_{q_2}^\dagger]e^{iq_3\cdot x}[\hat{b}_k, \hat{b}_{q_3}^\dagger] \tag{343}$$

$$= i\kappa\int d^4x e^{ix\cdot(k+k'-p)} = i(2\pi)^4\delta^4(p-k-k') \tag{344}$$

We have overall conservation of momentum as promised, and $\mathcal{M} = -\kappa$. In the exercises, you will compute $\tau$.

### 5.2.6 Contact Scattering $\varphi^4$ Interaction

How about scattering for a real field, mediated by $\mathcal{L}_I = -\lambda\frac{\varphi^4}{4!}$? There is a term to order the $\mathcal{H}_I$ expansion, so again:

$$\begin{aligned}\langle k, k'|\, S^{\text{tree}}\,|p, p'\rangle - 1 = \langle 0|\, a_k a_{k'}\frac{(-\lambda i)}{24}\int d^4x\int[d\vec{q}_1]&\left(a_{q_1}e^{-iq_1\cdot x} + a_{q_1}^\dagger e^{iq_1\cdot x}\right)\\
\cdot\int[d\vec{q}_2]&\left(a_{q_2}e^{-iq_2\cdot x} + a_{q_2}^\dagger e^{iq_2\cdot x}\right)\\
\cdot\int[d\vec{q}_3]&\left(a_{q_3}e^{-iq_3\cdot x} + a_{q_3}^\dagger e^{iq_3\cdot x}\right)\\
\cdot\int[d\vec{q}_4]&\left(a_{q_4}e^{-iq_4\cdot x} + a_{q_4}^\dagger e^{iq_4\cdot x}\right) a_p^\dagger a_{p'}^\dagger\,|0\rangle\end{aligned}$$

Every external state $a, a^\dagger$ will have to be commuted away, but there's so much choice! One such term gives

$$\frac{-\lambda i}{24}\int d^4x e^{ik\cdot x}e^{ik'\cdot x}e^{-ip\cdot x}e^{-ip'\cdot x} = -\frac{\lambda i}{4!}(2\pi)^4\delta^4(p+p'-k-k')$$

with no trace of 'which did we choose to contract with which?'. Combinatorics then say the other 23 options will combine for:

$$\langle k, k'| S^{\text{tree}} |p, p'\rangle = -i\lambda(2\pi)^4\delta^4(p + p' - k - k') \tag{345}$$

with the matrix element $\mathcal{M} = \lambda$. Again, quite a simple result after quite a lot of work.

## 5.3 Yukawa Theory - The Feynman Propagator

Back to our $\varphi, \phi$ theory we can compute scattering now, in particular:

$$\langle 0| \hat{a}_k\hat{b}_{k'} S^{\text{tree}} \hat{a}_p^\dagger \hat{b}_p^\dagger |0\rangle \tag{346}$$

We now have no non-zero term to $\mathcal{O}(\mathcal{H}_I)$, so to $\mathcal{O}((\mathcal{H}_I)^2)$

$$\langle 0| \hat{a}_k\hat{b}_{k'} \int d^4x \int d^4y \frac{i^2\kappa^2}{2}\Big[\Theta_{x^0-y^0}\varphi_x^I\phi_x^*\phi_x^I\varphi_y^I\phi_y^*\phi_y^I + \Theta_{y^0-x^0}\varphi_y^I\phi_y^{*I}\phi_y^I\varphi_x^I\phi_x^{*I}\phi_x^I\Big]\hat{a}_p^\dagger\hat{b}_{p'}^\dagger |0\rangle \tag{347}$$

We can find $\hat{a}$'s and $\hat{b}$'s to annihilate the out state (more than one now), but what are we to make of $a, a^\dagger$ in $\varphi_x, \varphi_y$? They have no-one but themselves to annihilate with. In particular, they contribute a factor:

$$\begin{aligned}\langle 0| T\{\varphi_I(x)\varphi_I(y)\} |0\rangle = &\Theta_{x^0-y^0} \langle 0|\varphi_I(x)\varphi_I(y)|0\rangle \\ &+ \Theta_{y^0-x^0} \langle 0|\varphi_I(y)\varphi_I(x)|0\rangle\end{aligned}$$

We have seen quite similar expressions, the first term is:

$$\Theta_{x^0-y^0} \int [d\vec{k}][d\vec{k}']e^{-ik\cdot x}e^{ik'\cdot y}[a_k, a_{k'}^\dagger] = \int \frac{d^3k}{(2\pi)^3 2E_k} e^{-iE_k(t-t')+i\vec{k}\cdot(\vec{x}-\vec{y})}\Theta_{x^0-y^0} \tag{348}$$

Now we can rewrite the Heaviside function as

$$\Theta(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\omega}{\omega - i\epsilon}e^{i\omega t} \tag{349}$$

which we integrate over the contours in fig. 21.

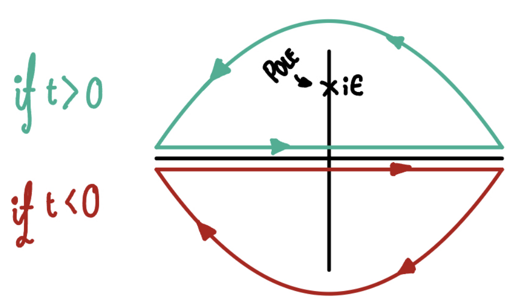



Figure 21: The contours over which we perform the time-integral in eq. (349): the upper contour for $t > 0$ and lower for $t < 0$. Note the pole which lies at $+i\epsilon$.

So we have that:

$$\frac{e^{-iE_k(x^0-y^0)}}{2E_k}\Theta_{x^0-y^0} = \frac{e^{-iE_k(x^0-y^0)}}{4\pi i E_k}\int d\omega \frac{e^{i\omega(x^0-y^0)}}{\omega - i\epsilon} \tag{350}$$

$$= \int \frac{d\omega}{2\pi i}\frac{e^{-i(x^0-y^0)(E_k-\omega)}}{2E_k(\omega - i\epsilon} \tag{351}$$

$$= \int \frac{dk^0}{2\pi i}\frac{e^{-ik^0(x^0-y^0)}}{(2E_k)(E_k - k^0 - i\epsilon)} \tag{352}$$

where $k^0 \equiv E_k - \omega$. Direct substitution gives

$$\int \frac{d^3k}{(2\pi)^3 2E_k} e^{-iE_k(t-t')+i\vec{k}\cdot(\vec{x}-\vec{y})}\Theta_{x^0-y^0} = \int \frac{d^4k}{(2\pi)^4 i(2E_k)}\frac{e^{ik\cdot(y-x)}}{E_k - k^0 - i\epsilon} \tag{353}$$

but for the second term in eq. (348), we flip $k^\mu \to -k^\mu$:

$$\int \frac{d^4k}{(2\pi)^4 i(2E_k)}\frac{e^{ik\cdot(y-x)}}{E_k + k^0 - i\epsilon} \tag{354}$$

and the sum of the two pieces is

$$\Delta_F(x-y) = \langle 0|T\{\varphi_I(x), \varphi_I(y)\}|0\rangle \tag{355}$$

$$= \int \frac{d^4k}{(2\pi)^4 i}\frac{e^{-ik\cdot(x-y)}}{2E_k}\left(\frac{1}{E_k - k^0 - i\epsilon} + \frac{1}{E_k + k^0 - i\epsilon}\right) \tag{356}$$

$$\boxed{\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 - i\epsilon}e^{-ik\cdot(x-y)}} \tag{357}$$

with $k^2 = (k^0)^2 - \vec{k}^2$. This is Feynman's propagator, central to pertubation theory.
Pushing ahead, we find several terms, for one of them, using $\Delta_F(x-y) = \Delta_F(y-x)$,

$$-\frac{\kappa^2}{2}\int d^4x d^4y e^{ix(k+k')}\Delta_F(x-y)e^{-iy(p+p')} \tag{358}$$

$$= -\frac{\kappa^2}{2}\int d^4x d^4y e^{ix\cdot(k+k'-q)}\frac{id^4q}{q^2 - M^2 - i\epsilon}e^{iy\cdot(q-p-p')} \tag{359}$$

$$= \frac{d^4q}{(2\pi)^4}\frac{(i\kappa)^2}{2}\frac{i}{1^2 - M^2 - i\epsilon}(2\pi)^4\delta^4(k+k'-q)(2\pi)^4\delta^4(q-p-p') \tag{360}$$

$$= (2\pi)^4\delta^4(k+k'-p-p')\frac{(i\kappa)^2}{2}\frac{i}{(p+p')^2 - M^2 - i\epsilon} \tag{361}$$

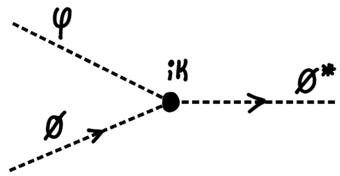
which once more conforms to the form we assumed. Possibilities we have not written are $\hat{a}_k, a_p^\dagger$ commuting in the same $\mathcal{H}_I(x)$, and a degeneracy of commuting onto $x$ or $y$, so that the final answer

$$-i\mathcal{M} = (i\kappa)^2\frac{i}{(p+p')^2 - M^2 - i\epsilon} + (i\kappa)^2\frac{i}{(p+k)^2 - M^2 - i\epsilon} \tag{362}$$

### 5.4 First Look at Feynman Diagrams

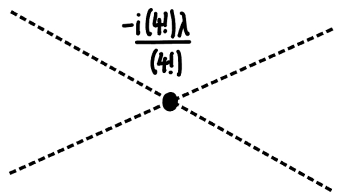
All the cases we have seen lead to simple expressions after laborious derivations, could there be an easier way to compute? Let us draw and assign:

$$\mathcal{L} = \kappa\, \varphi \phi \phi^* \quad \Rightarrow$$




(i)

$$\mathcal{L} = -\frac{\lambda}{4} \varphi^4 \quad \Rightarrow$$



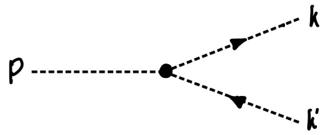

$$\Rightarrow \quad \frac{i}{q^2 - m^2 - i\epsilon}$$

(ii) Add $(2\pi)^4\delta^4(\sum p_{\text{in}} - \sum p_{\text{out}})$ at each vertex.

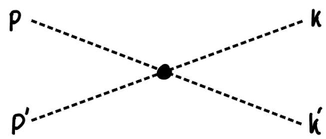
(iii) Integrate over external line's momenta $\int \frac{d^4q}{(2\pi)^4}$

We can now make diagrams for each of the processes we've looked at

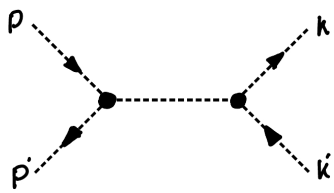
| $\xrightarrow[\textbf{time}]{}$ | **S-Contribution** |
|---|---|



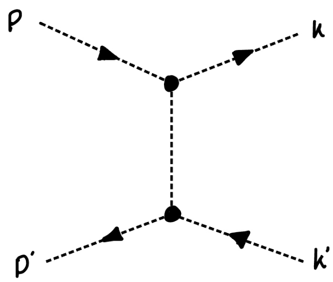

$$i\kappa(2\pi)^4\delta^4(p-k-k')$$




$$-i\lambda(2\pi)^4\delta^4(p+p'-k-k')$$




$$\begin{gathered} i\kappa(2\pi)^4\delta^4(p+p'-q) \\ \times \int \tfrac{d^4q}{(2\pi)^4}\,\tfrac{i}{q^2-M^2-i\epsilon} \\ \times(i\kappa(2\pi)^4\delta^4(q-k-k')) \end{gathered}$$




$$\begin{gathered} i\kappa(2\pi)^4\delta^4(p-q-k) \\ \times \int \tfrac{d^4q}{(2\pi)^4}\,\tfrac{i}{q^2-M^2-i\epsilon} \\ \times\left(i\kappa(2\pi)^4\delta^4(q+p'-k')\right) \end{gathered}$$

# 6 E: Interactions and the Perturbative Expansion [P.S. 4.2&4.3&7.2, S 6.1&7.A, Z III.3, T 3.3&3.7]

Eager as we were to see a way from theory to experiment, we have encountered some of the elements that make up any computation like vertices or Feynman propagators. Further, we have seen how the invariant matrix element lives up to its name; after the flurry of $a$, $a^\dagger$'s cancelling each other, rather simple, Lorentz-covariant results are obtained.

We now approach the perturbative expansion in a systematic way, which will allow us to understand further down the road the new phenomena that shows up beyond tree-level like: field renormalisation, running coupling constants or the fact that the vacuum is not empty.

Define, for real $\varphi$:

$$\varphi_I(x) = \varphi^\dagger(x) + \varphi^-(x) \tag{363}$$

$$\varphi^\dagger(x) = \int [d\vec{k}] e^{-ik\cdot x} a_k, \qquad\qquad \varphi^- = (\varphi^+)^\dagger \tag{364}$$

## 6.1 Wick's Theorem

We have seen how Feynman's propagator shows in our S-matrix element computation, and the time-ordering disappears. It would be computationally nice if this is true for more complicated processes. First, let's compute the two-field case:

$$\begin{aligned}
T\{\varphi_x^I \varphi_y^I\} &= \Theta_{x^0-y^0}(\varphi_x^+ + \varphi_x^-)(\varphi_y^+ + \varphi_y^-) + \Theta_{y^0-x^0}(\varphi_y^+ + \varphi_y^-)(\varphi_x^+ + \varphi_x^-) && (365a)\\
&= \Theta_{x^0-y^0}(\varphi_x^+\varphi_y^+ + \varphi_x^-\varphi_y^- + \varphi_x^-\varphi_y^+ + \varphi_y^-\varphi_x^+ + [\varphi_x^+, \varphi_y^-]) && (365b)\\
&= \Theta_{y^0-x^-}(\varphi_x^+\varphi_y^+ + \varphi_x^-\varphi_y^- + \varphi_y^-\varphi_x^+ + \varphi_x^-\varphi_y^+ + [\varphi_y^+, \varphi_x^-]) && (365c)\\
&=: \varphi_x^I \varphi_y^I : + \langle 0|T\{\varphi_x^I\varphi_y^I|0\rangle && (365d)\\
&=: \varphi_x^I \varphi_y^I : + \Delta_F(x-y) && (365e)
\end{aligned}$$

the two-field product decomposes into a c-number, $\Delta_F$, which we computed, the rest being normal ordered (hence cancelling between the vacuum). This relation is true for higher products. Consider the three-field product with $x_1^0 > x_2^0, x_3^0$

$$\begin{aligned}
\varphi_1^I T\{\varphi_2^I\varphi_3^I\} &= (\varphi_1^- + \varphi_1^+)(:\varphi_2^I\varphi_3^I: + \Delta_{23}) && (366a)\\
&=: \varphi_1^-\varphi_2^I\varphi_3^I : + : \varphi_2^I\varphi_3^I : \varphi_1^+ + [\varphi_1^+, : \varphi_2^I, \varphi_3^I :] + : \varphi_1^I : \Delta_{23} && (366b)\\
&=: \varphi_1^I\varphi_2^I\varphi_I^3 : + [\varphi_1^\dagger, \varphi_2^-] : \varphi_3^I : + [\varphi_1^+, \varphi_3^-] : \varphi_2^I : + \Delta_{23} : \varphi_1^I && (366c)
\end{aligned}$$

where $\Delta_{ij} = \delta_F(x_i - x_j)$. This is true for $x_1^0 > x_2^0, x_3^0$. The general expression adds up $x_2^0, x_3^0$ being the latest time and all combine into:

$$\begin{aligned}
T\{\varphi_1^I, \varphi_2^I, \varphi_3^I\} &=: \varphi_1^I\varphi_2^I, \varphi_3^I : + \overline{\varphi_1^I\varphi_2^I}\varphi_3^I + \varphi_1^I\overline{\varphi_2^I\varphi_3^I} + \overline{\varphi_1^I\varphi_2^I\varphi_3^I} && (367a)\\
&=: \varphi_1^I\varphi_2^I\varphi_3^I : + : \varphi_3^I : \Delta_{12} + : \varphi_1^I : \Delta_{23} + : \varphi_2^I : \Delta_{13} && (367b)
\end{aligned}$$

This generalisation builds on the $n-1$ product of fields to obtain the $n$ product, which we can then "write" as

$$
\begin{aligned}
T\{\prod_i^n \varphi^I(x_i)\} = L\prod_i^n \varphi_i^I : + \sum_{i<j}\Delta_{ij} : \frac{\prod_k \varphi_k^I}{\varphi_i^I\varphi_j^I} : \\
+ \sum_{\text{pair of pairs}} \Delta_{ij}\Delta_{kl} : \frac{\prod_m \varphi_m^I}{\varphi_i^I\varphi_j^I\varphi_k^I\varphi_l^I} : \\
+ ... \\
=: \prod_i^n \varphi_i^I : + (\text{All possible contractions})
\end{aligned}
$$

contractions being between fields that don't commute, so if we have a complex field, Wick's theorem reads as

$$T\{\phi_1^I\phi_2^{I*}\phi_3^I\phi_4^{I*}\} =: \phi_1^I\phi_2^{I*}\phi_3^I\phi_4^{I*} : + \phi_1^I\phi_2^{I*}\phi_3^I\phi_4^{I*} \tag{368a}$$

$$+ \phi_1^I\phi_2^{I*}\phi_3^I\phi_4^{I*} + ... \tag{368b}$$

The theorem also applies for coinciding spacetime points & it's of use to us to compute the time-ordered products. If we go back to our example Eq. 346, we needed:

$$T\{\phi_x^{I*}\phi_x^I\varphi_x\phi_y^{I*}\phi_y^I\varphi_y^I\} = ... + : \phi_x^{I*}\phi_x^I\phi_y^{I*}\phi_y^I : \Delta_{xy} \tag{369}$$

which allows us to separate internal/virtual particles, and external, out states.

$$\langle 0|\, \hat{a}_k\hat{b}_{k'} T\{\frac{(i\kappa)^2}{2}\int d^4x \int d^4y \varphi_x^I\phi_x^{I*}\phi_x^I\varphi_y^I\phi_y^{I*}\phi_y^I\}\hat{a}_p^\dagger\hat{a}_p^\dagger\,|0\rangle \tag{370}$$

$$= \frac{(i\kappa)^2}{2}\int d^4x d^4y \Big( \langle 0|\hat{a}_k\hat{b}_{k'} : \phi_x^{I*}\phi_x^I\phi_y^{I*}\phi_y^I : \hat{a}_p^\dagger\hat{b}_p^\dagger|0\rangle\Delta_{xy} \tag{371}$$

$$+ \langle 0|\hat{a}_k\hat{b}_{k'} : \phi_x^{I*}\phi_x^I\phi_y^{I*}\phi_y^I : \hat{a}_p^\dagger\hat{b}_p^\dagger|0\rangle\Delta_{xy} \tag{372}$$

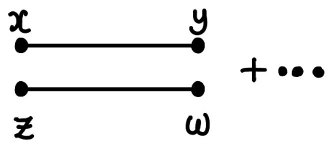
$$+ \langle 0|\hat{a}_k\hat{b}_{k'} : \phi_x^{I*}\phi_x^I\phi_y^{I*}\phi_y^I : \hat{a}_p^\dagger\hat{b}_p^\dagger|0\rangle\Delta_{xy} \tag{373}$$

$$+ \langle 0|\hat{a}_k\hat{b}_{k'} : \phi_x^{I*}\phi_x^I\phi_y^{I*}\phi_y^I : \hat{a}_p^\dagger\hat{b}_p^\dagger|0\rangle\Delta_{xy} \Big) \tag{374}$$

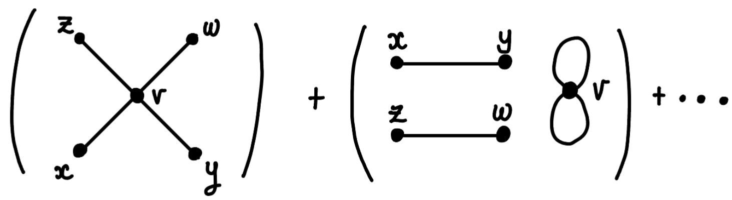
These two are identical contributions since $\Delta_{xy} = \Delta_{yx}$. As we did for the S-matrix, we can introduce Feynman diagrams for the Wick theorem & position space.

$$\langle 0|T\{\phi_I^x \phi_I^y\}|0\rangle = \Delta(x-y) = \tag{375}$$


$$\langle 0|T\{\phi_I^x \phi_I^y \phi_I^z \phi_I^w\}|0\rangle = \tag{376}$$

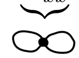



$$\langle 0|T\{\phi_I^x \phi_I^y \phi_I^z \phi_I^w (\phi_I^v)^4|0\rangle = \tag{377}$$




What are we to make of the disconnected piece? The answer is cancel it out. The diagram  is a vacuum fluctuation that we can't observe in scattering.

$$\langle 0|U_I(-\infty,\infty)|0\rangle = 1 - i\int d^4x \frac{\lambda}{8} \underbrace{\Delta^2_{xx}} + ... \tag{378}$$



## 6.2 Symmetry Factors

Some loop diagrams have numerical factors that cannot be extracted from Feynman rules alone. We call these factors **symmetry factors** and they can be determined by the Wick theorem. However, given the widespread use of Feynman rules we've come up with formulas to work them out. The symmetry factor $S$ is given by:

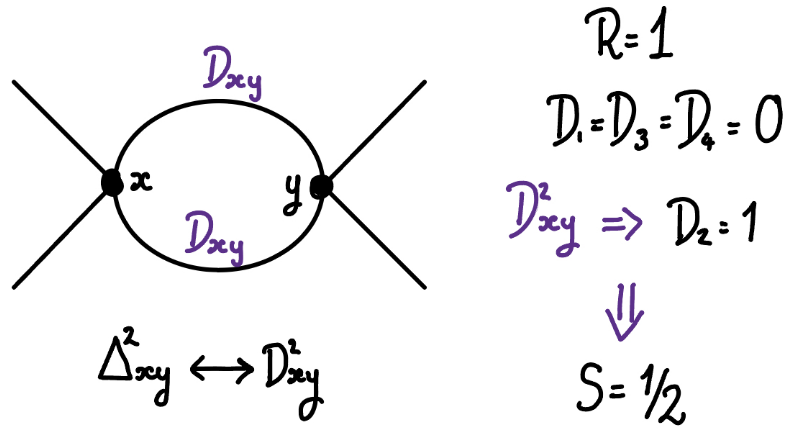
$$S = \frac{1}{R} \cdot \left(\frac{1}{2}\right)^{D_1} \cdot \left(\frac{1}{2!}\right)^{D_2} \cdot \left(\frac{1}{3!}\right)^{D_3} \cdot \left(\frac{1}{4!}\right)^{D_4} \tag{379}$$

where:

- $R$ is the number of ways to permute the internal indices and produce an identical set of propagators
- $D_1$ is the number of pairs of propagators of the form $D^2_{aa}$
- $D_2$ is the number of pairs of propagators of the form $D^2_{mn}$ plus the number of factors of the form $D^1_{aa}$ (N.B. the factor $D^2_{aa}$ contributes 1 to $D_1$ and 2 to $D_2$)
- $D_3$ is the number of sets of propagators of the form $D^3_{mn}$
- $D_4$ is the number of sets of propagators of the form $D^4_{mn}$

### 6.2.1 Example

Here is an example diagram with symmetry factor 1/2, which we calculate by noting that there are two propagators of the form $D_{\mathrm{xy}}$ so $D_2 = 1$, and that we can switch x and y once to get the same propagators so $R = 1$.




## 6.3 Green's Functions

Following on Wick's theorem steps, we can now put in a collection of objects: the all-order terms that enter scattering Green's functions, which we define as:

$|\Omega\rangle, |\lambda\rangle$ vacuum & "excited" states of interacting theory

$$G^{(n)}(x_1, x_2, ..., x_n) \equiv \langle\Omega|T\{\phi_H(x_1)\phi_H(x_2)...\phi_H(x_n)\}|\Omega\rangle \tag{380}$$

where

$$\begin{aligned}\phi_H(t,\vec{x}) &= U^\dagger(t,0)\phi(\vec{x})U(t,0)\\ [U(t,0) = U_0(t,0)U_I(t,t_i)U_0^\dagger(t_i,0)] \quad & \\ &= U_I^\dagger(t,0)\phi_I(t,x)U_I(t,0)\end{aligned}$$

We want to show how this relates to our interacting fields & perturbation theory. Between any two fields, we have (using the shortform $U_I(t) = U_I(t,0)$)

$$\phi^1_H\phi^1_H = U_I^\dagger(t_1)\phi^1_I U_I(t_1)U_I^\dagger(t_2)\phi^2_H U_I(t_2) \tag{381}$$

whereas at the end of a chain

$$\begin{aligned}\phi_H(t)|\Omega\rangle &= U_I^\dagger(t)\phi_I(t)U_I(t)|\Omega\rangle & (382)\\ &= U_I^\dagger(t)\phi_I(t)\underbrace{U_I(t,0)U_I(0,-\infty)}_{U_I(t,-\infty)}U_I(-\infty,0)|\Omega\rangle & (383)\end{aligned}$$

we have

$$U_I(\infty,0)|\Omega\rangle = U_0^\dagger(\infty,0)U(-\infty,0)|\Omega\rangle \tag{384}$$

$$= U_0(0,-\infty)|\Omega\rangle \tag{385}$$

where we have used that $|\Omega\rangle$ is the vacuum of the full theory with 0 energy. Next, we introduce the identity as the sum over *free states*:

$$U_0(0,-\infty)\left(|0\rangle\langle 0| + \sum_n |n\rangle\langle n|\right)|\Omega\rangle \tag{386}$$

$$|0\rangle\langle 0|\Omega\rangle + e^{i``\infty''E_n}|n\rangle\langle n|\Omega\rangle \tag{387}$$

Using the Riemann-Lebesgue theorem, given that the $\sum_s$ is really $\int[dk]$, we have all excited states cancelling [See Tong section 3.7]. The same holds on the left for $\langle\Omega|$, so we get:

$$\langle\Omega|\phi_H^1\phi_H^2...\phi_H^n|\Omega\rangle = \langle\Omega|0\rangle\langle 0|U_I(\infty,t_1)\phi_I^1 \times U_I(t_1,t_2)\phi_I^2...\phi_I^n U_I(t_n,-\infty)|0\rangle\langle 0|\Omega\rangle \tag{388}$$

The time-ordering can be applied and it condenses notation as:

$$\rangle\Omega|T\{\phi_H^1...\phi_H^n|\Omega\rangle = |\langle\Omega|0\rangle|^2\langle 0|T\{\phi_I^1...\phi_I^n U_I(\infty,-\infty)\}|0\rangle \tag{389}$$

where time-ordering chops up & orders the integral in

$$U_I(\infty,-\infty) = e^{-i\int_{-\infty}^{\infty} dt' H_I(t')} \tag{390}$$

The last step is setting $\langle\Omega|\Omega\rangle = 1$. For the equation above with no fields

$$1 = \langle\Omega|\Omega\rangle = |\langle\Omega|0\rangle|^2\langle 0|T\{U_I(\infty,-\infty)\}|0\rangle \tag{391}$$

for a final formula:

$$\boxed{\langle\Omega|T\{\phi_H^1...\phi_H^n\}|\Omega\rangle = \frac{\langle 0|T\{\phi_I^1...\phi_I^n U_I(\infty,-\infty)\}|0\rangle}{\langle 0|U_I(\infty,-\infty)|0\rangle}.} \tag{392}$$

We have achieved 2 things:

(i) Related the Green function to the interaction picture evolution operator & fields. This formula does resemble somewhat our S-matrix tree level expression.

(ii) We have factored out disconnected vacuum diagrams.

To see (ii) more clearly, let's do an example.

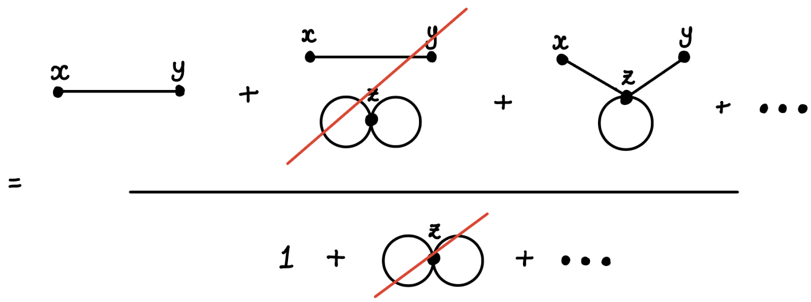
### 6.3.1 Contact Scattering $\phi^4$ Interaction

With our example, let's write

$$\langle\Omega|T\{\phi_H^x\phi_H^y\}|\Omega\rangle = \frac{\langle 0|T\{\phi_I^x\phi_I^y \exp\left(-i\int d^4z\mathcal{H}_I\right)\}|0\rangle}{\langle 0|\{\exp\left(-i\int d^4z\mathcal{H}_I\right)\}|0\rangle} \tag{393}$$

$$= \frac{\langle 0|T\{\phi_I^x\phi_I^y\left(1 - i\int d^4z\mathcal{H}_I + \mathcal{O}(\mathcal{H}^2)\right)\}|0\rangle}{\langle 0|\{\left(1 - i\int d^4x\mathcal{H}_I + \mathcal{O}(\mathcal{H}^2)\right)\}|0\rangle} \tag{394}$$

$$= \frac{\langle 0|\left(T\{\phi_I^x\phi_I^y\} - i\int d^4z\phi_I^x\phi_I^y\frac{\lambda}{4!}\left(\phi_I^z\right)^4 + ...\right)|0\rangle}{\langle 0|\{\left(1 - i\int d^4z\frac{\lambda}{4!}\left(\phi_I^z\right)^4 + ...\right)\}|0\rangle} \tag{395}$$




(396)

Though I'm cancelling "pictures" and asking you to believe me, you will do it yourself explicitly in problem sheet 4. Now for point (i), we can have a look at the tree-level Green function

$$\langle\Omega|T\{\phi_H^1\phi_H^2\phi_H^3\phi_H^4\}|\Omega\rangle = \Delta_{12}\Delta_{34} + (\text{combinations}) \tag{397}$$

$$\underbrace{-i\int d^4x\lambda\Delta_{1x}\Delta_{2x}\Delta_{3x}\Delta_{4x}}_{-i\lambda\int d^4x\left[\prod_i\int\frac{d^4q_i ie^{-iq_i(x_i-x)}}{(2\pi)^4(q_i^2-m^2)}\right]} + (\lambda^2) \tag{398}$$

$$= \left[\prod_i\int\frac{d^4q_i ie^{-iq_ix_i}}{(2\pi)^4(q_i^2-m^2)}\right](-i\lambda)(2\pi)^4\delta^4(\sum_i q_i) \tag{399}$$

This looks like the Fourier transform of the S-matrix element we found, but with a momentum-space propagator for each external leg. When we extend this relation to higher-orders, we obtain the LSZ reduction formula.

## 6.4 LSZ Reduction Formula

We've seen hints that an S-matrix element could be written to all orders in our expansion in terms of correlation functions, themselves Lorentz invariant. The derivation of such connection is what we'll do here. First, the Fourier transform of a Green's function for the Heisenberg field:

$$\tilde{G}^{(n)}(\{p\}) = \prod_k^n\int d^4x_k e^{ip_k\cdot x_k}\langle\Omega|T\left(\prod_l^n\phi_H(x_l)\right)|\Omega\rangle \tag{400}$$

where $p_k^0$ is a free variable. Let's pull a field to the left:

$$\begin{aligned}\langle\Omega|U(x^0)\phi(\vec{x})U(x^0,t_f)U(t_f,y^0)\phi(\vec{y}) &= \langle\Omega|e^{-i\vec{p}\cdot\vec{x}}\phi(0)e^{i\vec{p}\cdot\vec{x}}U(x^0,t_f)\sum_s|s\rangle\langle s|U(t_f,y^0)\\ &= \langle\Omega|\phi(0)\sum_s|s\rangle\langle s|e^{i\vec{q}_s\cdot\vec{x}}e^{-i(x^0-t_f)E_s}...\\ &= \sum_s\langle\Omega|\phi(0)|s\rangle e^{-iq_s\cdot x}\langle s,t_f|U(t_f,y^0)...\end{aligned}$$

The Fourier transform does it's job as

$$\int_{t_f}^{\infty}dx^0\int d^3x e^{ix(p-q_s)}\sum_s\langle\Omega|\phi(0)|s\rangle\frac{d^3q_s}{(2\pi)^3 2E_s} = \int_{t_f}^{\infty}\frac{e^{ix^0(p^0-E_S)}dx^0}{2E_s}\sum_s\langle\Omega|\phi(0)|s\rangle \tag{401}$$

$$= -\frac{e^{it_f(p^0-E_s)}}{2iE_s(p^0-E_s)}\sum_s\langle\Omega|\phi(0)|s\rangle \tag{402}$$

Next, we assume the only non-vanishing overlap of $\langle\Omega|\phi(0)|s\rangle$ is with a one-particle asymptotic state and define $Z\equiv\langle\Omega|\phi(0)|s\rangle$. We note that there is a pole $p^0=E_s$.

$$\tilde{G}^{(n)}\left(\{p_i\}\right)\xrightarrow{p^0\to E_s}\frac{i}{p^2-m_p^2}\sqrt{Z}\langle s,t_f|U(t_f,y^0)\phi_H(y^0)... \tag{403}$$

The last factor is a true asymptotic state, evolved to $t_f$, times the rest of the fields. This procedure can be repeated for the other fields, to be left with an out-state & in-state connected by $U$, $\langle\text{out},t_f|U(t_f,t_i)|\text{in},t_i\rangle$. This is the true scattering matrix to all orders, & it is given by the LSZ formula:

$$\boxed{\tilde{G}^{(n)}\left(\{p_i\};-\{k_l\}\right)\xrightarrow[k_l^0\to E_k]{p_i^0\to E_p}\left[\prod_i\frac{i\sqrt{Z}}{p_i^2-m_p^2}\right]\left[\prod_l\frac{i\sqrt{Z}}{k_l^2-m_p^2}\right]\times S_{\{k\}\to\{p\}}} \tag{404}$$

where $m_f$ is the physical mass, which differs from $m$ as given in the free action. It is rather a simple outcome that asymptotic & free states differ just by normalisation $Z$ and mass. The S-matrix is hence the "residue" in all external momenta. As in complex analysis, this means that the full function $\tilde{G}$ may and generally contain other terms.

## 6.5 Two-Point Function

We need first in our expansion to a given order, then $Z, m_p$. These appear in the 2-point Green's function which can be written, using translation invariance, as:

$$\tilde{G}^{(2)}(p,k) = (2\pi)^4\delta^4(p+k)\tilde{G}(p^2) \tag{405}$$

$$\tilde{G}\xrightarrow{p^0\to E_p}\frac{iZ}{p^2-m_p^2}\qquad\left[\text{Note the 1-pt only has one pole in } p^2\right] \tag{406}$$

We can compute this order-by-order, & for a fixed order even sum a countable $\infty$.

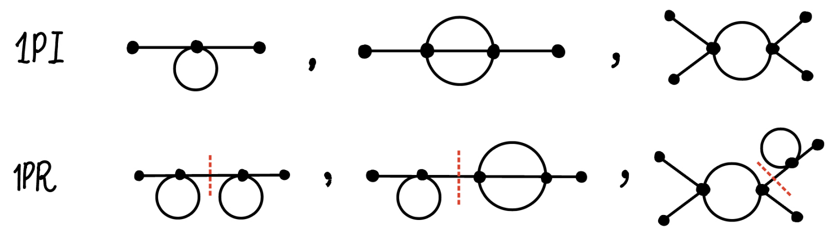
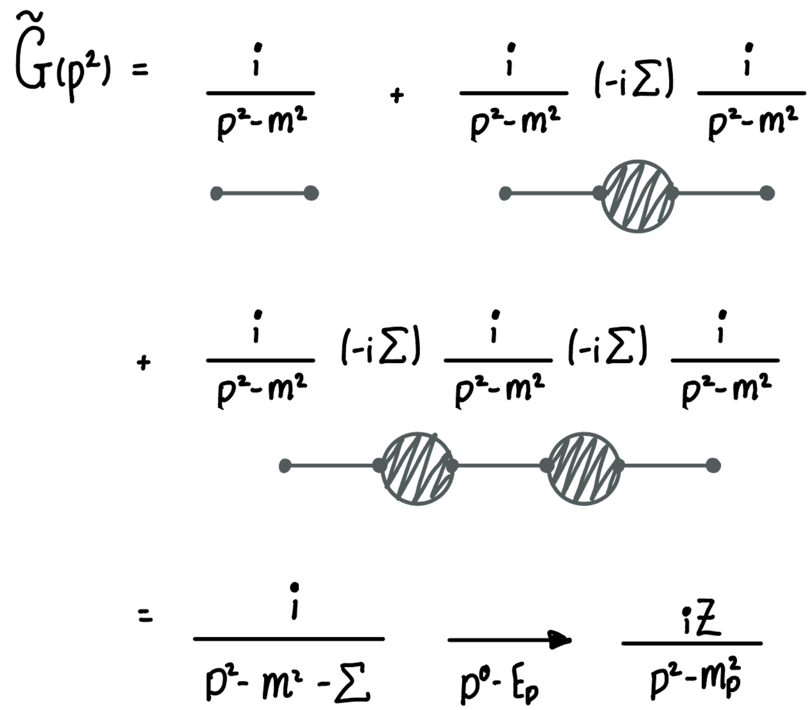



Figure 22: One particle irreducible (1PI) and one particle reducible (1PR) Feynman diagrams, on which the red dashed line shows the internal line which could split the diagram into two disconnected pieces.



$$\tilde{G}(p^2) = \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(-i\Sigma)\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(-i\Sigma)\frac{i}{p^2-m^2}(-i\Sigma)\frac{i}{p^2-m^2} = \frac{i}{p^2-m^2-\Sigma} \xrightarrow{p^0 \to E_p} \frac{iZ}{p^2-m_p^2}$$

where $\Sigma$ is the blob  representing all one-particle-irreducible (1PI) diagrams, and is called the self-energy. But what is 1PI? It's actually easier to define the opposite first:

| | |
|---|---|
| One particle reducible diagram | There exists at least one internal line whose absence (or cutting) would split the diagram into 2 disconnected pieces. |

Let's use $\lambda\varphi^4$ again for illustration in fig. 22 and the sum of these diagrams is the invariant matrix element, for a "1 to 1" process. So here:

$$-i\Sigma = \int \frac{d^4q}{(2\pi)^4}\frac{(-i\lambda)}{2}\frac{i}{q^2-m^2} + \mathcal{O}(\lambda^2) \tag{407}$$

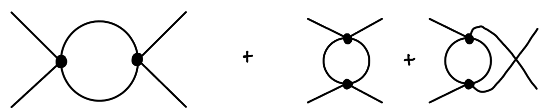
## 6.6 External Legs Corrections & Amputated Diagrams

Once we have the corrections to the two-point function, we can save ourselves some work at higher-point functions. Consider again $\lambda\varphi^4$ and the diagrams: DAIGRAMS These corrections we have computed already, so we can focus on the rest of the contributions, stripping out the external legs propagators. Then the LSZ formula reads:

$$\boxed{S_{\{p\}\to\{k\}} = (\sqrt{Z})^n \tilde{G}^{(n)}_{\text{amputated}}(\{p\}\to\{k\}) \qquad (408)}$$

$G_{\text{amputated}}$ can be computed as the invariant matrix elements in momentum space with Feynman rules, e.g.

$$G^{(\lambda)}_{\text{amp.}} = \int \frac{d^4q}{(2\pi)^2}\frac{(i\lambda)^2}{2}\frac{i}{(q^2-m^2)}\frac{i}{(q+p_1+p_2)^2-m^2} + ... \tag{409}$$



(410)

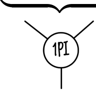
## 6.7 Effective Action & One Particle Irreducible

Lastly, we apply the 1PI definition on higher-point amplitudes, and define:

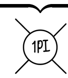
$$\begin{aligned} -i\Sigma &= \text{Sum of 2-point 1PI diagrams} \\ i\Gamma^{(3)} &= \text{Sum of 3-point 1PI diagrams} \\ &... \\ i\Gamma^{(n)} &= \text{Sum of n-point 1PI diagrams} \end{aligned}$$

With these pieces, we can build higher-point greens functions, take

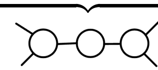
$$G^{(3)}_{\text{amp.}} = i(2\pi)^4\delta^4(\sum p)\underbrace{\Gamma^{(3)}(\{p_i\})} \tag{411}$$



$$G^{(4)}_{\text{amp.}} = (2\pi)^4\delta^4(\sum p)\cdot\Big\{i\underbrace{\Gamma^{(4)}(\{p_i\})} \tag{412}$$



$$+ i\underbrace{\Gamma(p_1,p_2,-p_1-p_2)\tilde{G}(p_1+p_2)i\Gamma^{(3)}(p_1+p_2,p_3,p_4)} \tag{413}$$



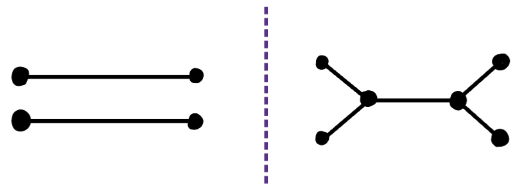
$$+ (p_3 \leftrightarrow p_4) + (p_2 \leftrightarrow p_4)\Big\} \tag{414}$$

where these expressions are exact, the expansion is within $\Gamma^{(n)}$. $\Gamma^{(n)}$ can be computed not only with diagrams, but also with functional methods in the path integral formulation. This, however, goes beyond our course.

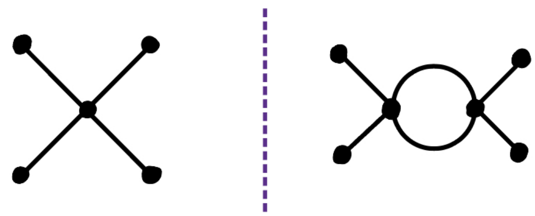
## 6.8 Diagram Zoology

It's good to collect here some of the diagram terminology:

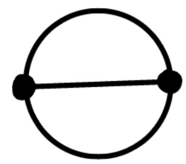
Connected/Disconnected



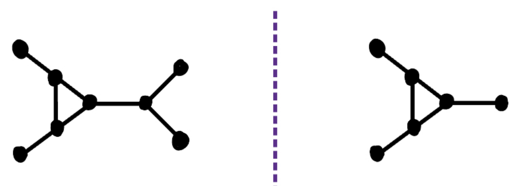
Tree Level/Loop Level



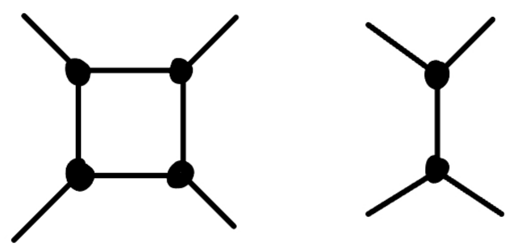
Vacuum Diagrams



One Particle Reducible/One Particle Irreducible



Amputated



# 7 F: Fermions

## 7.1 Lorentz group representations

What are the Lorentz group representations? A clue is in angular momentum being contained in it. This tells us that we're missing something between $\phi$ and $A^\mu$.

So what is $\sqrt{A^\mu}$?

$$\begin{aligned}
\text{Define :} \quad & \gamma^\mu = \gamma^0\,,\gamma^1\,,\gamma^2\,,\gamma^3\,, \qquad 4\times 4 \text{ complex matrices} && (415)\\
\text{such that :} \quad & \{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{I}_{4\times 4} && (416)\\
& (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 && (417)
\end{aligned}$$

A representation of the Lorentz group in this space is $\psi$ with

$$\psi \to \psi' = e^{i\sigma_{\mu\nu}\omega^{\nu\mu}/2}\psi \qquad \sigma_{\mu\nu} = \frac{i}{4}\left[\gamma^\mu,\gamma^\nu\right]\,, \tag{418}$$

and you can show that $(\sigma^{\mu\nu})^\dagger = \gamma^0\sigma^{\mu\nu}\gamma^0$.

We put forward the following

$$\text{proposition} \qquad \psi^\dagger\gamma^0\gamma^\mu\psi \text{ is a vector} \tag{419}$$

and proceed to prove it

$$\begin{aligned}
(\psi^\dagger\gamma^0\gamma^\mu\psi)' =& \psi^\dagger e^{-i\gamma^0\sigma\omega\gamma^0/2}\gamma^0\gamma^\mu e^{i\omega\sigma/2}\psi && (420)\\
\delta_\omega(\bar\psi\gamma^\mu\psi) =& \frac{1}{2}\psi^\dagger\left(-i\gamma^0\sigma\omega\gamma^\mu + \gamma^0\gamma^\mu i\sigma\omega\right)\psi && (421)\\
=& \frac{i}{2}\bar\psi\left(-\sigma\omega\gamma^\mu + \sigma\omega\gamma^\mu + \omega^{\beta\alpha}[\gamma^\mu,\sigma_{\alpha\beta}]\right)\psi && (422)\\
=& \frac{i}{2}\bar\psi\omega_{\beta\alpha}\frac{i}{2}\left(\eta^{\mu\alpha}\gamma^\beta - \eta^{\mu\beta}\gamma^\alpha - \eta^{\mu\beta}\gamma^\alpha + \eta^{\mu\alpha}\gamma^\beta\right)\psi && (423)\\
=& \omega^\mu{}_\alpha\bar\psi\gamma^\mu\psi && (424)
\end{aligned}$$

## 7.2 Dirac's field

What we have encountered, once promoted to space-time dependent, is Dirac's field $\psi(x)$. Let's build its quadratic action. Lorentz invariance sets constraints that leave us with two terms at this order

$$S = \int d^4x\left(i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi\right) \tag{425}$$

where $\bar\psi = \psi^\dagger\gamma^0$ and I encourage you to check that the m term is invariant.

The EoM is

$$\begin{aligned}
\frac{\delta S}{\delta\bar\psi} \quad \to \quad \left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)\psi =& 0 && (426)\\
\left(i\gamma^0\partial_t + i\vec\gamma\vec\nabla - m\right)\psi =& 0 && (427)
\end{aligned}$$

that is Dirac's equation.

Let's count degrees of freedom, $\psi$ is a 4-vector acted upon by $\gamma$ matrices so you might then count 4 real ($\psi$) and 4 imaginary $\psi$ degrees of freedom. However when counting degree's of freedom we clook at phase space:

$$\Pi_\psi = i\bar{\psi}\gamma^0 = i\psi^\dagger \tag{428}$$

which tells us there are half as many d.o.f. as you might think, 4.

The Hamiltonian density that follows is

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \partial_t \psi}\partial_t \psi - \mathcal{L} = -i\bar{\psi}\vec{\gamma}\vec{\nabla}\psi + m\bar{\psi}\psi \tag{429}$$

It is this operator that will give us the spectrum of the Quantum theory.

The Dirac field has then 4 d.o.f. We know that a spin 1/2 particle has two states, so there's two types of particles in $\psi$.

Let's stay at the classical level & solve for the EoM which will give us the ansatz for the quantum free-Hamiltonian-evolved-field, i.e. interaction picture field. Our ansatz

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\psi}(p) \tag{430}$$

so that the equation of motion

$$(\gamma^\mu p_\mu - m)\tilde{\psi}(p) = 0 \tag{431}$$

This algebraic eq. needs a vanishing determinant for any non-trivial solution to exist

$$\det(\gamma \cdot p - m) = (m^2 - p^2)^2 = 0 \tag{432}$$

There are 4 solutions to this eq.

$$(2\times) \quad p^0 = \sqrt{\vec{p}^2 + m^2} \qquad\qquad (2\times) \quad p^0 = -\sqrt{\vec{p}^2 + m^2} \tag{433}$$

We know that a spin 1/2 particle has two spin states, here we have two of those, one with negative $p^0$. Remember this is not an issue since we are looking at the field, not the wave-function. We treat this as we did for the complex field:

$$\psi(x) = \int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2)\left(A_s(p)u_s(p)e^{-ipx}\Theta(p^0) + B_s(p)^* v_s(p)e^{ipx}\Theta(p^0)\right) \tag{434}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p}\left(A_s(p)u_s(p)e^{-ipx} + B_s(p)^* v_s(p)e^{ipx}\right) \tag{435}$$

Where we have introduced $v_s, u_s$ as the vectors in Dirac space that

$$u_s \qquad (\gamma \cdot p - m)u_s = 0 \qquad p^0 > 0 \tag{436}$$

$$(p^0\gamma^0 - \vec{p}\vec{\gamma} - m)u_s = 0 \tag{437}$$

$$v_s \qquad (\gamma \cdot p + m)v_s = 0 \qquad p^0 > 0 \tag{438}$$

$$(p^0\gamma^0 - \vec{p}\vec{\gamma} + m)v_s = 0 \tag{439}$$

$$\tag{440}$$

So the trick is to interpret a particle with momenta $-p$ for $p^0 < 0$. We'll see if it works at the quantum level.

As eigenvectors, these satisfy orthogonality:

$$u_s^\dagger u_{s'} = 2E_p\delta_{ss'} \qquad v_s^\dagger v_{s'} = 2E_p\delta_{ss'} \qquad u_s^\dagger v_{s'} = 0 \tag{441}$$

& as Lorentz group reps

$$\bar{u}_s u_{s'} = 2m\delta_{ss'} \qquad \bar{v}_s v_{s'} = 2m\delta_{ss'} \tag{442}$$

$$\sum_s u_s\bar{u}_s = (\gamma p + m) \qquad \sum_s v_s\bar{v}_s = (\gamma p - m) \tag{443}$$

Here's a spinor in explicit form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \begin{array}{c} \sigma^\mu = (1, \vec{\sigma}) \\ \bar{\sigma}^\mu = (1, -\vec{\sigma}) \end{array} \tag{444}$$

$$u_s(p) = \begin{pmatrix} \sqrt{p\sigma}\chi_s \\ \sqrt{p\bar{\sigma}}\chi_s \end{pmatrix} \qquad v_s(p) = \begin{pmatrix} \sqrt{p\sigma}\tilde{\chi}_s \\ -\sqrt{p\bar{\sigma}}\tilde{\chi}_s \end{pmatrix} \tag{445}$$

we parametrise $\chi$ by the spin $\vec{S}$ on the rest frame as

$$\vec{S}\vec{\sigma}\chi_s = 1/2\chi_s \qquad \tilde{\chi}_s = \epsilon\chi_s^*\chi_s \qquad = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2e^{i\phi} \end{pmatrix} \tag{446}$$

with $\theta, \phi$ polar coordinates for the spin and

$$\sqrt{p\sigma} = \frac{(E_p+m) - \vec{\sigma}\vec{p}}{\sqrt{2(E_p+m)}} \qquad \sqrt{p\bar{\sigma}} = \frac{(E_p+m) + \vec{\sigma}\vec{p}}{\sqrt{2(E_p+m)}} \tag{447}$$

## 7.3 Chirality and irreducible representations

We have not written down the smalllest spin 1/2 representation yet; a Dirac spinor is reducible. Define

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad [\gamma_5, \sigma^{\mu\nu}] = 0, \quad \gamma_5^2 = 1 \tag{448}$$

This means, $\gamma_5$ $\sigma^{\mu\nu}$ can be simultaneously diagonalised, but $\gamma_5$ splits representations with projectors $P_L, P_R$

$$P_L \equiv \frac{1-\gamma_5}{2} \qquad P_R \equiv \frac{1+\gamma_5}{2} \tag{449}$$

$$\psi_L \equiv P_L\psi \qquad \psi_R \equiv P_R\psi \tag{450}$$

with $P_LP_R = 0$, $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L + P_R = 1$.

Given $P_L$ $P_R$ are projectors and the can be diagonalised simulatneously with Lorentz generators, $\sigma^{\mu\nu}$, $\psi_R$ and $\psi_L$ are closed under boosts and rotations

$$\psi_L' = P_Le^{i\omega\sigma/2}\psi = e^{i\omega\sigma/2}P_L\psi = e^{i\omega\sigma/2}\psi_L \tag{451}$$

and the same holds for $\psi_R$. Why didn't we start from the smallest representation? For Dirac mass term we need both left and right, as you can show yourself. One can still have mass with say $\psi_L$ only, but that is a Majorana fermion.

## 7.4 Dirac's Field Quantisation

We have used our experience of quantising a complex Klein-Gordon field to line up the classical theory so that quantisation should follow easily. As usual we do

and before we go any furhter let's look at the Hamiltonian

For our harmonic oscillator H first there must be a cancellation on

which does cancel since span a orthogonal basis.

For the rest, however

## 7.5 Spin-statistics

A consequence of the anti-commutation relations is the statistics as deduced from a two-state system. Take a scalar

but now for a Dirac field

The exchange of any Dirac particle implies a negative sign. We have discovered fermions of course!

In particular the above means

You can't put two fermions with the same spin on the same state. I we go back to a finite volume

For a boson, we can put as many particles as we like in the state but for a spin 1/2 frmions we can only put two: spin up and spin down.

## 7.6 Parity and Charge-conjugation

Parity does flip space's sign $x$ so a scalar field and derivative

$$\phi' \tag{452}$$

where we used.

In fact all 4-vectors will pick a sign and so should, which means

If our theory is parity-invariant , i.e. our action stays the same. Then

Charge conjugation. We exchange particles for anti-particles by

## 7.7 Pertubation theory for fermions

At the centre of our perturbative expansion in Lorentz invariant formulae was Dyson's formula introducing time-ordering $T$ and Wick's theorem to get rid of it.

We'll have to revise time ordering for fermions. When we put any fermion past another we picka a sign so now (all fields are interaction picture but we ommit)

$$T\psi \tag{453}$$

In particular

$$T\psi \tag{454}$$

and this also generalises to a Wick theorem for fermions.

Nonetheless, Dyson's formula time-orders the interaction Hamiltonian without any sign, is this fermionic Wick theorm of any use to us?

It is cause the Hamiltonian is a 'bosonic' operator; consider an interaction of two different fermions and a scalar

$$yukawa \tag{455}$$

A term in Dyson's formula is

$$T \tag{456}$$

Now suppose we evaluate it in between enough quanta to commute the fields away (e.g.) since we have arranged the rest in the right order, we would get

$$\langle 0| \tag{457}$$

which is the fermionic time ordering. THis is not a proof but this elementary result can be used for it.

**Feynman propagator**

Let's evaluate the above two point correlator then [with normalisation]

$$\langle 0|T\left(\psi_\alpha \bar{\psi}_\beta\right)|0\rangle = \Theta_{x^0-y^0} \tag{458}$$

or sometimes written as

**Causality**

Another concern is causality where we used the commutation relations for a scalar field. What saves the day in this case is that all observables have an even number of fermion fields. The way D. Tong puts it " no detector goes to minus itself when you rotate it 360"

Take for illustration the fermionic density $\psi^\dagger(x)\psi(x)$

$$[\psi^\dagger(x)\psi(x), \psi^\dagger(y)\psi(y)] \tag{459}$$

In computing this the following general relation is useful

$$[A, BC] = [A, B]C + B[A, C]$$

$$[A, BC] = \{A, B\}C - B\{A, C\}$$

Which helps break down the initial commutator into a factor $\psi$ whose vanishing for ensures causality is preserved

**Feynman Rules for fermions**

This will be the factor associated with an internal fermion line. For external states

$$\langle 0|\psi a^\dagger_{p,s}|0\rangle = e^{-ipx}u_s(p) \qquad \langle 0|\psi a^\dagger_{p,s}|0\rangle = e^{-ipx}u_s(p) \tag{460}$$

$$\langle 0|\psi a^\dagger_{p,s}|0\rangle = e^{-ipx}u_s(p) \qquad \langle 0|\psi a^\dagger_{p,s}|0\rangle = e^{-ipx}u_s(p) \tag{461}$$

Finally consider the extension of our Yukawa theory for fermions

$$\mathcal{L}_I = -y\varphi(x)\bar{\psi}(x)\psi(x) \qquad \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\mathcal{L}_I = -y\delta_{\alpha\beta} \tag{462}$$

We can collect all these in momentum-space Feynman rules

As we did before, we impose 4-momentum conservation at every vertex and integrate over internal momenta, but also sum over all spinor indexes to obtain a Lorentz invariant result.

An example

$$\bar{u}uf\frac{i}{(p_1 - p_3)^2 - m_\varphi^2}\bar{v}v \tag{463}$$

## 7.8 QED

The theory with the most precise tested productions goes by the name of Quantum Electro-Dynamics (QED). It is a gauge theory, which we have not discussed so far, so let's begin by giving a gauge transformation

$$\mathrm{U}(1)_{\mathrm{em}} \qquad \vec{A} \to \vec{A}' = \vec{A} + \nabla\alpha \qquad \Phi \to \Phi' = \Phi - \frac{\partial \alpha}{\partial t} \tag{464}$$

Then

$$\vec{E}' = -\nabla\Phi' - \partial_t \vec{A}' = \vec{E} + \nabla\partial_t\alpha - \partial_t\nabla\alpha = \vec{E}$$

The same holds for the magnetic field, so our Lagrangian $\varepsilon_0/2(E^2 - c^2B^2)$ is invariant and we've found a local symmetry $\mathrm{U}(1)_{\mathrm{em}}$ with $\alpha(x)$ the space-time dependent transformation parameter.

This type of symmetry is such that one could choose to give a phase to a field around the origin $\alpha(0) = \alpha_0$ bu not at some other point $\alpha(0,1,0,0) = 0$.

As it stands however this theory has no matter and the Noether current associated with the symmetry vanishes ().

Let's introduce a Dirac field $\psi(x)$. If our symmetry is U(1) then it seems sensible to assume

$$U(1)_{\mathrm{em}} : \qquad \psi \to \psi' = e^{-i\alpha/e}\psi \tag{465}$$

yet this would make the Lagrangian

$$\mathcal{L}'_\psi = \bar{\psi} e^{i\alpha/e}\left(i\gamma^0 \frac{\partial}{\partial ct} + i\vec{\gamma}\nabla - m\right) e^{i\alpha/e}\psi\nabla\alpha J \tag{466}$$

$$= \mathcal{L}_\psi + \frac{1}{e}\left[\bar{\psi}\gamma^0\psi\frac{1}{c}\frac{\partial\alpha}{\partial t} + \bar{\psi}\vec{\gamma}\psi\nabla\alpha\right] \tag{467}$$

These terms we could cancel out with $\bar{\psi}\gamma^\mu\psi A_\mu = \bar{\psi}\gamma^0\psi\Phi/c - \bar{\psi}\vec{\gamma}\psi\vec{A}$, recall $A_\mu = (\Phi/c, -\vec{A})$ for a Lagrangian as

$$\tag{468}$$

with $D \equiv \partial - ieA$. In natural units (which we now take to include)

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{469}$$

with

$$\frac{e^2}{4\pi} \simeq \frac{1}{137} \qquad F_{\mu\nu}\psi = \frac{i}{e}\left[D_\mu, D_\nu\right]\psi \tag{470}$$

This is the most general renormalisable action one can build compatible with Lorentz and gauge invariance.

It does have charged particles with

$$-\frac{\alpha}{e^2}J^\mu_{\mathrm{em}} = \delta_\alpha A_\nu \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu} + \delta\psi\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} = \frac{\alpha}{e}\bar{\psi}\gamma^\mu\psi \tag{471}$$

The gauge invariance removes one d.o.f. of $A^\mu$ on top of the non-dynamical $A^0$ so when quantised this theory presents

- $\mathbf{e}^-$ electrons with spin 1/2 mass m, charge -e
- $\mathbf{e}^+$ positrons with spin 1/2 mass m, charge e
- $\mathbf{A}_\mu$ photons with spin 1 (but two states only) mass 0, charge 0

## Feynman Rules

### Interaction picture fields

$$\Psi_I(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_s \left( a_{k,s} u_s(k) e^{-ikx} + b^\dagger_{k,s} v_s(k) e^{ikx} \right) \tag{472}$$

$$A^I_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_\lambda \left( c_{k,s} \varepsilon^{(\lambda)}_\mu(k) e^{-ikx} + h.c. \right) \qquad \varepsilon_\mu(k) k^\mu = 0 \tag{473}$$