Introductory Field Theory — Problem Set 1

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1. **Lagrangian and Hamiltonian mechanics**

   Consider the following Lagrangian for two real degrees of freedom $q_1(t), q_2(t)$ given in terms of $Q \equiv q_1 + iq_2$

   $$L = \frac{1}{2}\dot{Q}^*\dot{Q} + \frac{m^2}{2}Q^*Q - \frac{\lambda}{4}(Q^*Q)^2$$

   where λ, m are positive constants and Q^* the complex conjugate of Q.

   i) Perform a Legendre transform to obtain the Hamiltonian; you can use either $\dot{q}_1, \dot{q}_2$ or $\dot{Q}, \dot{Q}^*$ as your variables.

   ii) Write the quantity $\mathcal{O} = q_1\dot{q}_2 - q_2\dot{q}_1$ in terms of $Q, \dot{Q}, Q^*, \dot{Q}^*$.

   iii) Show that the quantity $\mathcal{O}$ is conserved by writing it in terms of position and momentum and taking its Poisson bracket with the Hamiltonian.

   iv) Find the solution to the equations of motion with

   $$Q(0) = \frac{m}{\sqrt{\lambda}}e^{i\pi/3}\,, \qquad \dot{Q}(0) = 0\,.$$

2. **Legendre transformation**(LP)

   Consider the function

   $$F(v) = \frac{\beta v^4}{4}$$

   a) Obtain the Legendre transform $G(w)$ with $w = F'(v)$.

   b) Evaluate $G'(F'(x))$ to determine the relation between G' and F'.

   c) Perform another Legendre transform now on $G(w)$ to obtain $K(z)$ with $z = G'(w)$.

   d) Compare $K(z)$ and $F(v)$ and use your result to determine how to go from the Hamiltonian to the Lagrangian.

3. **Lagrangian and Hamiltonian field mechanics**

   Study the dynamics of the following systems

   a) **Klein Gordon:** Starting from the Hamiltonian density

   $$\mathscr{H} = \frac{1}{2}\left(\Pi^2 + (c_\phi \nabla\phi)^2 + \alpha\,\phi^2\right)$$

   derive Hamilton's equations and show that they lead to Euler-Lagrange's equations (as derived in class).

   b) **Electromagnetism:** Derive Maxwell's equations from the Lagrangian density [you can use or derive yourself that $(\nabla \wedge \vec{V}) \cdot (\vec{W}) = \vec{V} \cdot (\nabla \wedge \vec{W}) +$ (boundary term) or $\nabla \wedge \nabla \wedge \vec{V} = \vec{\nabla}(\nabla \cdot V) - \nabla^2\vec{V}$]

   $$\mathscr{L} = \frac{\varepsilon_0}{2}\left((\vec{\nabla}\Phi + \partial_t\vec{A})^2 - c^2(\nabla \wedge \vec{A})^2\right) - \Phi\rho + \vec{A} \cdot \vec{J}$$

   Obtain the Hamiltonian density, then set $\rho = \vec{J} = 0$ and check if it can be given in terms of $E = -\nabla\Phi - \partial_t A$ and $B = \nabla \wedge A$ only. Using the constraint that follows from $\Pi_\Phi = 0$ on $\dot{\Pi}_\Phi$'s equation and integration by parts try again to obtain $\mathscr{H}(E, B)$.

c) **Complex, constrained field:**(LP) When the Hamiltonian depends on $\nabla\Pi$ we retain the two terms in Hamilton's equation for $\dot{\phi}$:

$$\dot{\phi} = \frac{\partial \mathscr{H}}{\partial \Pi} - \nabla \frac{\partial \mathscr{H}}{\partial \nabla \Pi}$$

Derive Hamilton's equations for

$$\mathscr{H} = \frac{-i\hbar \nabla\Pi \nabla\varphi}{2m}$$

and show that they lead to Euler Lagrange's equation

$$i\hbar \partial_t \varphi = -\frac{\hbar^2 \nabla^2 \varphi}{2m}$$

given $\Pi = i\hbar\varphi^*$.

d) **Acoustic phonon:**(LP) Find Euler-Lagrange's equation with the variational principle substituting $D \to D + \delta D$ inside all derivatives and neglecting boundary terms for the Lagrangian

$$\mathscr{L} = \frac{1}{2}\left[m\dot{D}^2 - K\left(\ell\nabla D\right)^2 + \frac{K}{4}\left(\ell^2\nabla^2 D\right)^2\right]$$

4. Complex Klein-Gordon Quantisation

Consider the Lagrangian density for a complex KG field

$$\mathscr{L} = \dot{\phi}^*\dot{\phi} - c^2\nabla\phi^*\nabla\phi - \frac{m^2c^4}{\hbar^2}\phi^*\phi$$

i) Compute the Hamiltonian treating ϕ and ϕ^* as independent variables.

ii) Check that the ansatz

$$\phi(\vec{x}) = \hbar \int [d\vec{k}] \sqrt{\frac{N_k}{2E_k}} \left(a_k e^{i\vec{k}\cdot\vec{x}} + b_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right)$$

$$\Pi(\vec{x}) = i \int [d\vec{k}'] \sqrt{\frac{N_{k'}}{2E_{k'}}} \left(E_{k'} a_{k'}^\dagger e^{-i\vec{k}'\cdot\vec{x}'} - E_{k'} b_{k'} e^{i\vec{k}'\cdot\vec{x}}\right)$$

with

$$[a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = (2\pi)^3 N_k \delta^3(\vec{k} - \vec{k}') \qquad [d\vec{k}] = \frac{d^3k}{(2\pi)^3 N_k}$$

provides a canonical commutation relation for ϕ and its canonical conjugate momentum.

iii) Put the ansatz in the Hamiltonian and in order to write it as a harmonic oscillator Hamiltonian (i.e. cancelling ab and $b^\dagger a^\dagger$ terms), deduce the relation $E_k^2 = E_k^2(m, \hbar\vec{k}, c)$.